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CONTENTS

Page

LUIGI AMOROSO: Vilfredo Pareto 1

J. TINBERGEN: On the Theory of Business-Cycle Control 22

K.-G. HAGSTROEM: Pure Economics as a Stochastical Theory 40

H. W. ROBINSON: The Equilibrium Price in a Perfect Intertemporal Market . 48

FRANK H. KNIGHT: On the Theory of Capital: In Reply to Mr. Kaldor . . 63

A. L. BOWLEY: Note on Professor Frisch's "The Problem of Index Numbers" . 83

RAGNAR FRISCH: The Double-Expenditure Method 85

ANNOUNCEMENTS, NOTES, AND MEMORANDA

Program of the Annecy Meeting, September 11–15, 1937 91

Officers and Council for 1938 91

Election of Fellows, 1937 92

Fellows of the Econometric Society, January, 1938 95

Summary of Accounts, 1930–1937 96

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Vilfredo Pareto
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VILFREDO PARETO*

By LUIGI AMOROSO

THE MARQUIS VILFREDO PARETO, was born July 15, 1848 at Paris, where his father, Raffaele Pareto, an old Genoese patrician, and a partisan of Mazzini, was living in voluntary exile. He spent the earlier years of his life in France, but he carried on all of his studies in Italy, where he returned in 1858, receiving his doctor's degree in engineering in 1869 from the Polytechnic Institute of Torino. He practiced his profession for about twenty years and was manager of the Iron Works of San Giovanni Valdarno and later general manager of the Italian Iron Works. A man of the world, he frequented, during that period, high society of Florence and, in particular, the salon of Signora Emilia (Peruzzi), for whom, as well as for her husband Ubaldino, he always retained a deep affection. The nature of his activities naturally brought him face to face with the fundamental problems of political economy and particularly with that of protective tariffs. As an industrial manager, he did not limit himself to the consideration of his own individual case, but boldly entered the field as the paladin of the doctrine of free trade, in the name of which he attacked the protectionist policy of the time, denouncing its spiritual poverty and its servility to private interests. Later, in the maturity of his genius, in the quiet of Céligny, he abjured his former writings of polemic character, not because his judgment on men and things had changed, but because his theoretic attitude had changed, and wider was the angle from which he judged facts.

He was still living in Florence, when the reading of *Elementi di Economia Pura* by Maffeo Pantaleoni attracted him to abstract speculation. It was at first a complementary occupation of his life: then it became by degrees ever more absorbing, until it caused him to give up all other activity. From 1890 to 1905 he contributed assiduously to the *Giornale degli Economisti* and gave a rigorous scientific foundation to the theories of mathematical economics, which had already been introduced into the science by Cournot, Jevons, and Walras and formed later his general theory of economic equilibrium. He accepted the offer of the Canton of Vaud, Switzerland, and became in 1893 the successor of Walras in the chair of Political Economy at the University of Lausanne.

* Based on an article by the author in *Enciclopedia Italiana*, Vol. xxvi, p. 326.

Giving up teaching in 1906, he retired to Céligny in the Canton of Geneva, where, in the quiet of "Villa Angora," he lived the last period of his life, entirely devoted to study. This was the most fruitful period of his scientific activity, during which his fundamental works were published: the *Manuale di Economia Politica* and the *Trattato di Sociologia*. He still continued to follow the economic and political changes of the time, and study and meditation brought him gradually to a spiritual attitude, which was antithetical to the one he held at the beginning, so that the old liberal who at first tended to radicalism, if not to socialism, was gradually transformed into the most profound and vigorous critic of socialism and of democracy.

The fundamental qualities in the mental formation of Pareto are mathematical knowledge and humanistic culture. Pareto, an engineer, in addition to Italian, wrote French in a literary manner, read fluently English, Latin, and Greek. He had translated the Greek Anthology for the mere pleasure of doing a linguistic exercise and in his youth had even attempted a comparative analysis of the language of Saint Paul and the Attic dialect. The thoroughness of his culture in this field is one of the most graceful charms of his works, where poets, prose writers, philosophers, metaphysicians, historians, all the classics of Greek, Latin, Italian, French, and English literature are garnered to confirm or to refute this or that conclusion which seemed to follow from the examination of statistics or from the discussion of theories.

Born of a French mother, he loved France, not only from filial affection, but because his thoroughly Latin genius attracted him to France. He was grateful to Switzerland which gave him hospitality, but he was always, and above all, Italian and he loved Italy with infinite tenderness. In his adopted land he preserved his Italian nationality, just as his father, the Marquis Raffaele, had preserved his in the land of exile. And if he considered it proper that the works composed during the period he was teaching at Lausanne, and which reproduced the courses given in that University, should appear in the language of the young men for whom they were destined, he returned—just as soon as he was free from the duties of official teaching—to the language of his father. It is symptomatic to recall, in this connection, that the date of his retirement from his teaching position, precedes by one year that of the publication of the *Manuale*.

The Governments of democratic Italy, before and after the war, paid no attention to him. The Fascist Government, in the very first months of its installation, made him Senator of the Kingdom, together with his great friend and contemporary, Maffeo Pantaleoni. The Fascist Government also asked him to be the delegate of Italy to the League of Nations at Geneva on the question of disarmament. "It is to be re-

gretted that the state of his health did not permit him to accept,"—so writes Bousquet, one of his biographers,—"for the world would have witnessed then a spectacle such as it had not been offered since the appearance of Bismark at the Bundestag at Frankfort."

Pareto died March 2, 1923 in the full luster of his genius and his remains repose in the little cemetery of Céligny.

The principal works of Pareto are, in chronological order, as follows: *Cours d'Économie Politique*, Lausanne, F. Rouge, 1896–97; *Systèmes socialistes*, Paris, Giard et Brière, 1902; second edition by the same publisher, 1926; *Manuale di Economia Politica*, Milano, Società Editrice Libraria, 1906; French translation, Giard et Brière, 1909; reprint of the French edition, 1927 (the French edition differs from the Italian in the Mathematical Appendix); *Trattato di Sociologia generale*, Firenze, Barbera, 1916; second edition 1923; French translation, published by Payot 1917–19; English translation, *The Mind and Society*, New York, Harcourt, Brace & Co., 1935; *Fatti e teorie*, Firenze, Vallecchi, 1920; *Trasformazioni della democrazia*, Milano, Corbaccio, 1921.

The *Cours* serves as an introduction to the *Manuale:* the *Systèmes* prepare and the last two small works complete the *Trattato di Sociologia.*

1. THE CURVE OF INCOMES

Pareto's mathematical skill appears first of all in the *Cours*, where is given the equation of the curve of incomes, which is, among all researches of inductive economics on the basis of statistics, the most elegant, if not indeed the most fecund.

This equation is $N = Ax^m$, where N represents the number of persons who have an income higher than x, while A and m are two constants, the first of which measures the amount of the total wealth, and the second, its distribution. The parameter m is almost a constant and in this invariability from country to country and from epoch to epoch, which later researches have confirmed, approximately, is the deep significance of Pareto's discovery. Wherefore the equation is not a mere statistical interpolation, but an instrument which allows us to penetrate deeply into the mechanism of the distribution of wealth.

2. ECONOMIC EQUILIBRIUM

Even more than in statistics, however, Pareto's mathematical skill shows its vigor in the construction of the general theory of economic equilibrium which is contained in the *Manuale* and still more in the French translation (*Manuel d'Économie Politique*), in which the mathematical appendix has been completely reworked.

Here is its fundamental concept.

Let us suppose that a certain number of independent parameters $q_1, q_2, \cdots, q_n$ represent the configuration of a certain system, and that a certain function $R(q_1, q_2, \cdots, q_n)$ has the following property: q_r being a generic variable, if to a positive increase Δq_r there corresponds for R a positive increase ΔR, the system moves in the direction in which all the q_r are positive; if, on the other hand, the increase ΔR corresponding to Δq_r is negative, the system moves in the direction in which all the q_r are negative; and, finally, if the increase ΔR corresponding to Δq_r is zero, the system does not move at all.

The configuration of equilibrium will be obtained, then, by writing the system of equations

$$\frac{\partial R}{\partial q_1} = 0, \quad \frac{\partial R}{\partial q_2} = 0, \cdots, \frac{\partial R}{\partial q_n} = 0.$$

A function R which has this property is called an *index function.*

These ideas are extremely general. They apply equally well to mechanical systems and to economic systems.

They have a still more general range. The form of the theories does not change, even if the meaning of the function R is changed; the essential is that it be an index of the direction in which it is anticipated that the system will move. If an index function of altruism could be found, substituting it for R, we should have the mathematical theory of altruism.

The simplest case is that of a single producer, the index function representing then the net profit of the production. By calling this profit F and the quantity sold q, F is a function of q and represents an index function in the sense indicated above. The configuration of equilibrium is then determined by the equation

$$\frac{dF}{dq} = 0,$$

which indicates that the producer aims at the maximum gain in money.

3. THE INDEX FUNCTION OF OPHELIMITY

The treatment of the case of a single consumer is equally simple, *if it is admitted that one can measure pleasure.* The quantity which represents such a measure in function of consumption is then the index function of the system.

The function which measures pleasure was called *ophelimity* and from the study of it pure economics was born with the works of Gossen, Jevons, Walras, Marshall, Edgeworth, Pantaleoni, and Fisher. The quantities that measure what at the beginning was called *final degree of utility, scarcity,* were nothing but the partial derivatives of ophelim-

ity. The determination of the position of economic equilibrium in the problems of exchange had been reduced, by the authors whom we have cited and by Pareto in the *Cours*, to the search for the relative maxima of the function of ophelimity.

In the *Manuale* it is recognized that it is not correct to speak of a measure of pleasure and hence not of the function of ophelimity; it is recognized, moreover, that ophelimity is not an essential idea. To construct the general theory of economic equilibrium it is sufficient to have functions (indices of ophelimity) which will increase when the pleasure increases and which will decrease when the pleasure diminishes. Finally in the *Manuale* the notion of pleasure is dropped entirely. Instead, we can assume a function which will serve to indicate the direction in which the movement is foreseen to happen.

Here is an analogy with the evolution of the concepts of *Rational Mechanics*. In Statics one began by considering a system of points and a system of forces applied to them. Supposing for simplicity the system free of constraints, the conditions of equilibrium would be expressed by saying that the resultant of all the forces applied should be zero. It was at once recognized that these same conditions were expressed more simply by saying that the positions of equilibrium corresponded to the maxima and minima of the potential function. Finally it was seen that the concept of forces could be dispensed with entirely. The position of equilibrium was determined just as soon as a function (index) was known, which, with the sign of its derivatives, would indicate the direction of the movement.

The concept of applied forces corresponds in Economics to the concept of final degree of utility. The potential function corresponds to the ophelimity.

4. EQUILIBRIUM AS A CONTRAST BETWEEN TASTES AND OBSTACLES

The actions developed by forces are fettered by *obstacles*.

For every individual the tastes of the others with whom he contracts are obstacles. If the quantity of goods must be divided among several persons, the fact that the quantity is fixed is decidedly an obstacle; if the good to be divided is produced, the fact that it can be obtained only by using other wares still constitutes an obstacle; likewise the fact that the good is not available in the place and at the time it is needed constitutes another obstacle. In general the meaning of obstacles is that economic goods are limited; that until violence and fraud, theft and donation are excluded, a thing cannot be had except by giving in exchange for it one of equal value pro tempore; that every product is the result of a certain combination of the factors of production in harmony with the laws of technics, as they are known pro tempore; that the

legal order and the economic organization fetter individual actions; and so on.

Exclusive of the introductory and complementary parts, the *Manuale* consists essentially of three chapters: in the first, a synthetic study is made of tastes, that is, the forces which impel to action; in the second, of the obstacles, that is, the constraints which are opposed to the tastes; in the third, are studied the configurations of equilibrium which arise from the contrast of those forces and of these constraints.

And what about production and exchange, consumption and distribution? There does not exist in real life a distinction in things corresponding to this distinction of words. There does not exist a problem of production, as distinct from a problem of distribution and vice versa, but all economic problems are included in the general conditions of equilibrium.

5. THE CRUX OF PARETO'S SYSTEM

Classical economics rested and still rests in the main on the assumption—implicit, even if not always explicit—that economic equilibrium represents a typical configuration around which the real configuration oscillates now in one direction and now in another; or, in short, that equilibrium is *an idea*, in the Platonic sense, of which reality may offer the more or less deformed image. Such a conception, forming the substratum of the antithesis between normal value and current value, as it was already set forth in *The Wealth of Nations* of Adam Smith, inspiring the greater part of the economic literature of the eighteenth century, and constituting even today the fundamental theme of contemporary theoretical investigations, is not surmounted in the *Manuale*. But Pareto noted even then the theoretical deficiency of the doctrine of equilibrium, in so far as the dynamic aspect is the essential, not the contingent of economic reality, and this latter is not polarized around an ideal configuration, but moves incessantly in an eternal change, under the action of external and internal forces, which bind the present to the past, the future to the present. The external forces—the only ones that are explicitly considered in the *Manuale*—represent applied forces, the tastes and the obstacles in the model of Pareto.

It is at this point that the crux of Pareto's system becomes apparent. The internal forces of the economic system are not susceptible of a theoretical representation as simple, elegant, and universal as is the case for the applied forces. They are not only, as for the material macrocosmic systems, forces of conservation, by which—to express it elegantly—the dead city dominates *through inertia* the living city; they are also directive forces or forces of impulsion, through which the living city *forms or attempts to form* the city of the future. The internal forces, therefore, are History, they are even Ethics and Politics, something

powerful, but vague and indistinct, which is not susceptible of mathematical representation; an expression of the freedom of the will, which does not allow itself to be enclosed in the meshes of a mechanical representation, and, because it is mechanical, determinist.

Pareto succeeds in his purpose, basing his theory on the fact that the movement of economic phenomena cannot be separated from that of political and social phenomena. Economic dynamics merges into Politics, or, to use Pareto's term, into Sociology.

6. THE INSTINCT OF COMBINATIONS

The structure of Pareto's Sociology rests on the following foundation: In human phenomena—and consequently in the theories devised to explain these phenomena—there is always a constant element and a variable element. The first is the manifestation of instincts, feelings, appetites, etc.; the second is constituted by logical and illogical reasonings which express the need for justifying, in a rational way, what has no rational origin. Thus, for example, in the doctrines which proclaim the existence of a moral law, the constant element is constituted by the postulation of certain norms of life; the variable element is constituted by the appeal to supernatural or metaphysical entities, in the name of which the norm is imposed.

Pareto calls the constant element *residue;* the variable element, *derivation* or *derivative.*

How are the residues classified? The world, the devil, and the flesh are, according to the Catholic doctrine, the obstacles in the way of eternal salvation. The corresponding categories of Pareto are: the residue of sociality, the residue of integrity, and the sexual residue. To these must be added then the need of manifesting one's own feelings by external acts, the instinct of combinations, and the persistence of the aggregates. So that, actually the classes of Pareto's residues are six.

Of special importance in Pareto's system are the instinct of combinations and the persistence of the aggregates.

The instinct of combinations—according to Pareto—is characteristic of the human species. With a box and a little ball a cat plays a whole day and has not an idea. The baby immediately puts the ball into the box. Behold—says Pareto—the instinct of combinations. This was and is a powerful promoter of civil progress. A very great number of phenomena arise from the inclinations to combine certain things.The learned man combines them according to certain norms, certain opinions, certain hypotheses; and we have *scientific combination* which gives rise to discovery and invention; the multitude make combinations guided by analogies, in most cases fantastic, puerile, and casual; and thus we have *vulgar combination.*

Vulgar combination is most of the time sterile in the sense that it has no other aim than that of satisfying the instinct of the one who makes it; thus it is in the game of lotto, in the magic arts, in the processes of sorcery. At other times, however, it borders on the scientific combination and gives rise to discoveries. Chemistry was born as a consequence of the efforts in search of the philosophers' stone; medicine in an analogous way. Any one who reads the *Natural History* of Pliny is struck by the infinite number of combinations attempted to cure diseases. So, the domestication of animals is the fruit of a process, centuries old, which had its origin in a vulgar combination, that is, a combination which had no other aim but that of satisfying an instinct. The same can be said of a thousand other discoveries, which are milestones in the history of human civilization, such, for example, as fire, boxes, metals, etc.

7. THE PERSISTENCE OF THE AGGREGATES

The persistence of the aggregates represents—according to Pareto—the residue, thanks to which certain combinations, once formed, become *stable* and a firm possession of individuals and of human society. They constitute a sort of aggregate of parts closely joined as in a single body, which finally acquires a personality similar to that of other real beings. The instinct which is opposed to the separation of these parts is precisely the persistence of the aggregates.

The chief way in which the persistence of aggregates shows itself is in the persistence of the relations of one man with other men and places. From it come *the feelings* (*sentiments*), *so-called, of the family, of property, of patriotism, of love of one's own language, of one's own religion, of the fields*, etc.; usually, there are added derivations and logical explanations which perhaps conceal the residue.

Living in a given milieu impresses on the mind certain concepts, certain modes of thinking and of acting, certain prejudices, certain beliefs, which are antithetical to those that are formed in other environments, and which by this antithesis are strengthened, persist, and acquire an objective existence. This is a general fact which gives rise to the distinction in social classes, in sects, in parties, with the innumerable forms in which the phenomenon is manifested concretely.

Also the cult of the dead is an expression of the persistence of the aggregates; a further proof of this is the fact that the materialists honor their dead just as the faithful believers do. The concept of the survival of the one who has died substantially rests on the feeling, most powerful in us, of the unity of a man in the flight of years. In reality the corporeal part of a man changes and the psychic part too; neither materially nor morally is an old man identical with himself when he was a

baby; still we admit that there is in them a unity which remains. He who is not frightened with words calls this unity *soul.* He who is frightened at them rejects the words, but he cannot reject the concept, because without it every man would be declared irresponsible for all that he has done in the past.

8. THE DEMOCRATIC OLYMPUS

Once constituted, the aggregate becomes something different from the simple superposition of the elements which constitute it. As the flock is something different from the mere sum of the sheep of which it is composed, so the aggregate becomes an entity in itself, and gives rise to an abstraction which persists, after having acquired individuality and a life of its own. In the same way was born the idea of the sacred rivers, for example, the German Rhine; the same is the origin of the goddess Annona, in whom the Romans personified the provisioning of the city. Similar is the origin of the apotheosis of the emperors; the emperor, whoever he was, personified the Empire, the regular administration, justice, the Roman peace; and those sentiments in no wise vanished because a man died and another took his place; the permanence of that aggregate was the fact; the apotheosis, one of the forms in which it was manifested.

Analogously the democratic nineteenth century personified "Liberty," who is, however, only one of the divinities that dwell in the Olympus of the religion of democracy. This, like all religions, has a theology in which the history of humanity is presented as the history of the conflict between a principle of evil, embodied in a cohort of divinities such as *Superstitition, Private Property, Capitalism,* etc., and a principle of good, embodied in another cohort of divinities, such as *Science, Democracy, Humanity, Liberty, Truth, Justice.* The ones and the others are true and proper *supernatural entities,* who, like Apollo and Venus, Minerva and Juno, descend to the battlefield for Troy and against Troy. The victory of the principle of good over the principle of evil constitutes *Progress.*

As in the case of the gods of Homer—continues Pareto—so also with the divinities of the democratic Olympus, their credit is now raised and now lowered. In the period which immediately preceded the French Revolution *Private Property* was in the front rank, on a level with *Superstition,* and Rousseau denounced it with vehement invectives. At the time of the revolution *Superstition* again reigned with many a follower such as kings, nobles, priests. Afterwards people turned to other theoretical speculations, and *Capitalism* succeeded *Private Property,* as Jove followed Saturn. *Blessed he who possesses the key of Knowledge!* Every phenomenon, present and future, is explained with

the magic word of Capitalism. It alone is the cause of misery, of ignorance, of bad customs, of thefts, of assassinations, of wars. It is no use to cite the example of savage people, who drag out their existence in perpetual war, for faith commands to believe that, without Capitalism, there would not be any wars. If there are the poor, the ignorant, the lazy, the wicked, the alcoholic, the insane, the dissolute, the thieves, the assassins, the conquerors, the blame is to be laid only on Capitalism. The reasoning with which this is proven is the usual one: *post hoc ergo propter hoc*. Society is *capitalistic*, hence its evils have their origin in *Capitalism*. Other reasons are added which in substance reduce themselves to asserting that, if men had of everything unto satiety, they would not commit evil acts or crimes in order to procure it; and since it is an article of faith that only *Capitalism* prevents men from having everything until satiety, it is proven that this entity (the capitalistic concept) is the cause of every evil act.

All this is not peculiar to the religion of democracy. All religions have a theology and all have an Olympus, and all fight and ought to fight to conquer the *universality of minds* and the unity of faith; to extirpate—with force—the disintegrating heresies. Thus indeed it was with the religion of democracy, which in the period of its splendor showed all the characteristics of all the other religions, intolerance being in the front rank.

9. THE SECULAR TREND OF HUMAN SOCIETY

Going on to study how the forces which have been considered are combined to determine social equilibrium, Pareto demonstrates that a first schema of social dynamics, with reference to secular trends, can be obtained by taking into account these two circumstances, namely: the proportion in which the residues of the instinct of combinations and of the persistence of the aggregates are present in the dominating class and in the dominated class; and the velocity of circulation of the elite, this being formed by individuals who occupy an eminent position politically, economically, in art, in moral and religious activity, in general in any one of the various forms of human activity. The velocity determines the degree of the so-called social flexibility.

In Pareto's mind, a society would be able to prosper when the circulation should not be too much facilitated, in order to assure stability; nor too much obstructed, in order to permit a certain degree of social exchange. He shows that, given a certain intermediate velocity between the two extreme limits, and a certain intermediate proportion of the two fundamental residues in the dominating class and in the dominated class, there must be a certain combination of these conditions which is the most favorable for assuring the prosperity of the State.

In general, in the governing class the residues of the instinct of combination ought to prevail over the residues of the persistence of the aggregates; the contrary should hold for the governed class; but in the long run, the men in whom prevail the residues of the instinct of combinations are weakened and the social circulation must, therefore, consent for new men to assume the position of the first.

In substance, a State declines, either because there is a lack of equilibrium between the instinct of combinations and the persistence of the aggregates, or because the circulation is too much accelerated or too much retarded. Sparta fell from the prevalence of the residue of the persistence of the aggregates; Athens, on the contrary, fell from the prevalence of the residue of the instinct of combinations.

If Athens, in her foreign policy, had to limit herself to the defense, while Macedonia could carry out a policy of expansion and conquest, the reason, according to Pareto, must be that Macedonia, much more than Greece, knew how to adjust herself to the proportion between the two residues, which better than the others secure the maximum of stability. Rome arrived still closer to this ideal position, and that explains the universal success of her foreign policy. The contrast between the two residues can explain all the Roman history. The last years of the Republic—for example—are characterized by the prevalence, in the governing class, of the residue of combinations; the revolutions of Sulla, of Cataline, of Caesar must indicate that the proportions which had been thus formed were not favorable. The governed class triumphed with Augustus and hence, with the Empire, there begins in Rome a perior in which the residue of the persistence of aggregates prevailed: such a prevalence assures at the beginning a period of prosperity, but it is at the same time the first origin of that social crystallization which leads later to the fall of the Empire.

The same cause—social rigidity—determines, according to Pareto, the fall of Venice. He finds that, at a certain moment in Venetian history, it seemed as if there were about to be instigated a certain circulation of the elite which would have secured the necessary social exchange. The condemnation of Marin Faliero and the subsequent exclusiveness of the dominant class were a germ which caused that this state, although it still had so many fortunate combinations to prosper for a long time, had a shorter life than Sparta, Athens, and Rome had had.

10. ECONOMIC AND SOCIAL CYCLES

The consideration of the proportion of the two fundamental residues and of social flexibility is sufficient to give an idea of the movement of human societies in the secular development, but it becomes inadequate when the field of vision is restricted in time, to embrace the events of

one or few generations. To gain in depth what is lost in extent, it is then necessary to enlarge the frame of the theoretical schema, adding to the residues and to the social flexibility the consideration of two other elements: the *derivations*, that is, the reasonings by which men seek to give a logical system to their life and to their beliefs; and the *interests*, which constitute the impulse of daily acts. Therefore, in more restricted limits of time, the movement appears to Pareto like the result of the actions and reactions of these four groups of factors:

(A) the residues,
(B) the derivations,
(C) the interests,
(D) the social flexibility.

The group (A) works powerfully on (B), (C), and (D), and this has been recognized by all who have proclaimed that *Ethics* is the foundation of the social order. Usages and customs, legal order, political and economic organization, literature, art, morality, religion, reflect powerfully the currents induced by this group. On the other hand, the currents that are induced by the group (B) on (A), on (C), on (D), are rather less potent and that explains the contempt which men of action have always displayed for *ideologies*—liberty, equality, fraternity, natural law, law of nations, etc.—and the facility with which a new ideology may sweep away the ancient one. It is, however, necessary, to guard against undervaluing, too much, the weight of such reactions. It is true that Christianity does not owe its victory to the Doctors nor the French revolution to the Philosophers, but it is true too that the theoretical elaborations of the Doctors of the fourth century and those of the Philosophers of the eighteenth were of great importance for re-enforcing, exalting, and purifying the nascent movement.

The contempt for ideologies created among the theorists of socialism the vogue of *historical materialism*, which in the structure of Pareto is systematized as the doctrine which formulates the reactions of the group (C) on (A), (B), and (D). Certainly the interests, as the expression of the need for daily bread, are powerful factors which act upon feelings, upon ideas, upon the social organization, and the currents which they induce are, as a rule, stronger than those that come from the brain, but not stronger than those that come from the heart. The error of historical materialism is not only in having substituted the part for the whole, but also in having asserted a *universal* predominance, when on the other hand, to the predominance of Economics over Logic ought to have been added its subordination to Ethics. It is evident that the word *Logic* must here be understood not in the Aristotelian and Baconian sense, as the theory of the syllogism and experi-

mental induction, but in Pareto's meaning of the theory of derivations, that is, of the logical and illogical reasoning with which the human species gives, or thinks it gives, a rational order to its own life.

Ethics, Logic, and Economics formulate theoretically, therefore, the actions which the elements, indicated above as (A), (B), and (C), exercise, the ones on the others and all of them on (D). The reactions which, vice versa, (D) induces on (A), (B), and (C) are those that happen on the inside of the social body, and act powerfully in modifying feelings, ideas, and economic organization. The accumulation of elements of elite in the lower strata of society, and of degenerate elements in the upper strata, is the eternal cause of all the revolutions. Aristotle, Machiavelli, and Vico are the classical authors who have traced the fundamental lines of these reactions; to the body of doctrine which reduces them to theory, the first of them, i.e., Aristotle, gave the name of Politics.

Concretely, the single actions and reactions are compounded and interfere in a thousand ways and from their contrast arises the social movement. Analyzing the single interferences, Pareto deduces the oscillatory character of this movement. The single oscillations constitute *the economic and social cycles* and form the warp on which are interwoven the vicissitudes of the history of human society.

11. SPECULATORS AND RENTIERS

Let us pause a moment on the reactions induced by group (C) and let us fix our minds, for example, on industrial protection. Its most important effects are as follows: rise in the prices of goods of consumption, increase in industrial production, greater difficulties for the increase of agricultural production, shifting of wealth from the classes with fixed income to those of variable income, acceleration of the movement of circulation of wealth and more generally in the circulation of the elite.

The effect of the shifting of wealth is the most important. To the class (*with variable income*) which becomes rich belong the entrepreneurs, the stockholders of industrial and commercial companies, the owners of houses and lands—where speculation in building exists—the speculators on the stock exchange, the bankers, and all the persons who are dependent upon them as workmen, employees, notaries, lawyers, engineers, politicians, etc. On the other hand, to the class (*with fixed or almost fixed income*) which is impoverished belong mere possessors of savings, depositors in the banks or savings banks, the annuitants, the pensioners, the holders of State securities or bonds of corporations, the owners of houses and lands where speculation does not occur, and also the employees, the workmen, the farmers, who are dependent upon

those persons. Pareto indicates by S (speculators) those belonging to the first class and by R (rentiers, i.e., possessing an income or rent) those who belong to the second class. Industrial protection, which enriches S to the detriment of R, is extolled by the former and condemned by the latter. The former (S) think *the collective utility is in the increase* of industrial production; the latter (R) hold that it is on the contrary in the *reduction of the prices* of consumption goods. Both rest their assertions on metaphysical reasoning or on verbal proofs, or on other arguments which belong to the dialectics of faith. In reality, it is the motive of individual interests which acts instinctively in determining the judgment of both groups of people.

The opposition between S and R is manifested—Pareto continues—in every controversy of political economy. When it is considered not in an abstract way, *sub specie aeternitatis*, but concretely, that is, *in relation to a given situation of fact*, every choice, in the economic field, involves, explicitly or implicitly, the following dilemma: is it advisable to aim for the stability of economic relations, or to favor, stimulate, encourage the creation of new situations? The S, adventurous and unscrupulous people, always decide for the second alternative, sure as they are that with genius and astuteness they will be able to turn the novelty to their own advantage. An opposite choice is made by the R, honest men and timid in the matter of economic initiative, in whom gifts of character predominate over those of technical and financial ingenuity, and who know by experience that, however things go, they will end by paying the expenses of every innovation.

In the struggle, as long as it remains on economic ground and is therefore fought with shrewdness, the S, who are rather more ready on the offense than the R are on the defense, usually win. But the R, in their turn, become predominant when they carry the battle to another terrain, to the terrain of politics, for example, and substitute for the weapon of shrewdness that of force.

In various forms, this economic struggle may appear in political garb, for example, as conflict between the divine right of kings and the right of the people, which is then, in substance, the opposition between the interests of the present generation and those of the future generation. *The interest of S many times*, not always however, *is contary to that of the future generation*, in so far as the classes which are easily enriched by the success of ingenious economic combinations rarely have the virtue of saving, and, earning liberally, they tend to consume even more liberally. On the other hand, the interest of the future generation is in the production of new capital, and this—*all other conditions being unchanged*—is the more intense, the greater the saving. It can happen, however, that the yield of the economic operations of S will

be large enough to permit an ample increase in consumption, without giving rise to a reduction in saving. In fact, it did so happen during the nineteenth century through the enormous increase of production brought in by great industry. But history shows that such a fortunate coincidence is exceptional and transitory: in general, *the expansion of consumption occurs to the detriment of saving*. In the very tendency of the increase of production and expansion of consumption to determine a decrease of saving, while—on the other hand—saving is the condition *sine qua non* for giving life to ingenious financial combinations, Pareto sees the maximum of internal *contrasts* which brings with it the so-called *industrial cycle*.

12. THE FREEDOM OF THE PRESS

Pareto analyzes at length the characteristics presented by the daily press in the so-called regime of *freedom of the press*.

He begins by observing that it would be manifestly excessive to assert that all the news and all the articles of the newspapers are always inspired in the defense of an interest; the respect for sentiments that are dominant or even merely widespread, plays here an important part, nay, a very important part. Especially the party press has an intuition, that is, a peculiar feeling for the life of the country; and in this it is inspired each day. This does not obviate, in many other cases, especially in the so-called journalistic campaigns, the prevalence of the motive of the interest of the one who owns the paper, or of the one who pays for it, or of the one who has any power at all over it. The thing is not always manifest and he who is a novice at journalistic skirmishing does not ordinarily discover the location of the bow from which the arrow was shot. When, about the middle of the past century, in the struggles which preceded, accompanied, and followed the enterprise of cutting through the isthmus of Suez, the English press attacked de Lesseps furiously, the ingenuous reader would never have thought that those fiery articles were inspired by de Lesseps himself. And yet, many years afterwards, he himself related and explained the fact, saying that the greater part of his expenses for propaganda in England were made up of sums paid to have himself attacked. The attacks—he explained to the Minister Olivier, who was naïvely astonished at it—are the means of attracting attention. They are forgotten at once and there remains only the memory of the name and of the thing attacked.

In the regime of the freedom of the press—Pareto continues—there are opposition papers paid by the governments; and socialist papers paid by high finance; and nationalistic papers in which a foreign Government, or even international Free Masonry, has a finger. However absurd the matter may seem, it is a fact; and it is explained by say-

ing that it may be sufficient for a Government or any power whatsoever for a paper not to touch certain keys at a certain moment, or for it to touch certain others; and when it touches them, to do so in a certain manner. Certain attacks, especially on account of the place whence they come, or on account of the manner in which they are conducted, are often more powerful than defense. Bismarck was a master of the art of buying the press in this way, at relatively small expense.

The strength of the newspapers is in the art of stirring up feelings, by using the reasonings which are proper to the dialectics of faith. Feelings must exist first, and that determines the limits of the power of the newspaper, which cannot go against them, but only make use of them for *its own purpose*. Provided it responds to a widespread feeling, any argumentation is good, whether it invokes a principle of authority, or makes an appeal to a metaphysical entity (idea) or even simply repeats verbal proofs. The sophism does not harm, it may even be useful; a simple demonstration is wanted which can be understood by all and be repeated indefinitely.

13. THE PARLIAMENTARY GAME

Like the daily press, the parliamentary game in a democratic regime is also the expression of private interests: in a more limited degree in the golden periods of parliamentarism, in a predominant degree in the periods of decadence.

In the period when Parliament represents truly a live and real force, the great debates which occur in the Chamber and the legislation resulting from them reflect above all the feelings which are disseminated in a large mass of the population and which are translated into a special intuition of moral life. Then the private interests are relegated to the second place. They do not, however, entirely disappear, as is shown by parliamentary inquiries, which occur in all epochs, but especially in the golden periods of parliamentary institutions, since, in these periods, the sense of responsibility of the men of politics is more awake to danger, and the reaction of public opinion more prompt. The inquiries show, now in a veiled way, now in all clearness, how, underneath the reasonings of the political men for pushing on this or that action, private interest is concealed.

When the parliamentary institutions are in the full vigor of their strength, the inquiries cut short the political career of the one who is blamed by them. In the periods of decadence, on the other hand, the case of a man severely censured by an inquiry returning to the ministry and becoming the master of the country is not rare. So, Pareto recalls the case of Rouvier, who, though implicated in France in the Panama scandal, could return to the Government. Thus, in England,

Lloyd George remained a Minister after the inquiry on the stock-exchange operations. And in Italy Giolitti, after the scandals of the "Banca Romana," came back as head of the Government and was, for more than a decade, the ruler of Parliament.

When parliamentary decadence is accentuated, the motive of private interest becomes the warp on which is woven all of parliamentary life. The Chamber is transformed into a new kind of *feudalism*, and the art of politics is wholly reduced to providing for the interests of the new lords (*elected*) and new vassals (*electors*). Parliamentary debate is the instrument to snare the favor of the public and divert its attention, and further still the weapon for simulating and defending, for striking and offending; all that with the consciousness or semiconsciousness that this brawl is, at bottom, something like a ceremony or a performance. All say that the speeches will not change a situation nor displace a single vote; still the speeches are made, and all orators and listeners feel, more or less consciously, that they are playing a part, like the actor in the theatre. These are words of Ciccotti, a former Italian deputy of the democratic party, and Pareto has made them his own.

14. ITALIAN POLITICS IN THE PERIOD IMMEDIATELY PRECEDING FASCISM

Returning to the distinction between *R* (*rentiers*) and *S* (*speculators*), let us recognize that in a regime of parliamentary decadence, the head of the Government is, in substance, like the head of the syndicate of *S*. He has the appearance of the power, of which they have the substance. This explains—Pareto continues—how for so many years a man like Depretis could be a master of the Italian Chamber. He was not the head of a victorious army—writes Pareto—he did not have the eloquence which stirs men, nor the authority that comes from great deeds, nor was he imposed by the Sovereign. Whence, then, came his great strength? There can be only one answer. He knew, in a masterful way, how to make use of feelings and of interests which were in the country; the protective tariff, the railway agreements, public works, the leases and banking disorders were the expression of this policy.

Crispi offered—according to Pareto—an interlude: a Government moving and acting according to a directive line of its own, in correspondence with its own intuition of the moral life of the country, and seeking to persevere in its way surmounting the obstacles which were interposed. But he did not have the skill to avail himself of the feelings existing in the country for putting into practice his own moral idea. He combated the socialists harshly. Nevertheless he was not able —as Mussolini later did—to disintegrate their party, to attract the masses to himself, to avail himself of the strong forces latent in them,

to transform these forces from elements of disintegration—such as they were—into elements of national cohesion. He did not succeed in uniting the *R* around himself, while the *S*, to whom he gave nothing to bite at, hated him. Moreover he had unfavorable conditions in the economic cycle which, during the time he held the Government, was moving towards a period of depression.

For Pareto, the contrast with his successor, Giolitti, is significant. No less than Depretis, Giolitti made himself the head of the syndicate of the "speculators," the protector of trusts, of monopolies, of public works, and, since the conditions of the economic cycle were favorable to him, he could make friends with the socialists, by subsidizing the co-operatives liberally, and satisfy the nationalists by making war in Libya. He was a master of the art of turning to account existing feelings. While he kept at bay the socialists and nationalists, he extended voting rights to frighten the bourgeoisie and become their protector, and, at the same time, to appear the paladin and the standard-bearer of the rights of the people. With the same idea he allowed strikes periodically to disturb the life of the country, but just as soon as he was aware that the measure was about to be heaped up and reaction was manifested in public opinion, he used repression energetically. He combated the clericals publicly and flattered them secretly. In short, there were no sentiments and interests in Italy of which he did not avail himself for his own end.

This man, however, never went beyond the aim of dragging out his political life from day to day. Never did Giolitti have an idea of his own to assert, never could he rise above the contingencies of the moment. While he gave the Ottoman Empire a serious shaking up with the Libyan war, he was not aware that he was preparing the Balkan war and, therefore, disturbing profoundly the equilibrium of Europe. He was not, therefore, preoccupied with preparing the military forces of the country, in view of future conflicts. Not only did he not increase military expenses, in order not to incur the opposition of the socialists, but he persisted—and bragged of it—in the policy of extending public works, of subsidizing the co-operatives, of broadening social legislation, all of which meant subsidies to the electoral masses. He made war and disguised its burdens, hiding the expenses of it among the folds of the accounts and by postponing the liquidation of them to the future. He covertly increased the public debt with the emission of long-term Treasury Bonds. All this was useful for the moment, because it pleased those who wanted war and those who were unwilling to bear the necessary consequences of it, but prepared the difficulties of the future.

When the problem of the European war, which involved the very life of the nation, and which could find no solution in the criterion of

life from day to day, was presented, he fell. He had a brief resurrection when, after the war was over, he blindly thought that its general problems were settled, and that he could return to the old methods and the old systems.

What happened later in Italy after his third fall was part of the historic cycle which takes its name from the Fascist revolution and constitutes the passion of our life. The Fascist revolution has been, at the same time, a revolution of men, a revolution of ideas, an overturning of pre-existent interests, and an overthrow of moral values. Its unity which wholly subsists in the midst of the variation of external circumstances offers a demonstration of the reactions of mutual dependence between the categories which Pareto has put as the basis of his theoretical system.

15. THE MYSTICS OF PARETO

The ancient Greek naturalists saw in Zeus the origin of all the phenomena of nature. Saint Augustine searched in the Sacred Writings for the proof of the existence of the antipodes. Galileo was the first to teach that Physics is to be conceived solely as the synthesis of facts. Pareto declares that he was the first to introduce the method of Galileo into Sociology.

He was mistaken. Spinoza, not to mention any other, had preceded him. The *Tractatus Politicus* of the great thinker of the seventeenth century opens with the declaration that the fault of the philosophers lies in considering human passions as vices into which men fall through their own fault: *Affectus quibus conflictamur, concipiunt philosophi veluti vitia, in quae homines sua culpa labuntur*, and herein lies the reason, because philosophers think they have attained the sum total of knowledge, when they have learned to dispense praise and blame. They conceive men not as they are in reality, but as they imagine them to be. *Homines namque non ut sint, sed ut eosdem esse vellent concipiunt*, whence it follows that they believe that they are making Ethics when they are making nothing but Satire; that they are making Politics when, instead, they are navigating in the realm of Chimera and of Utopia. To this point of view the great thinker opposes his own point of view. *Cum igitur ad politicam applicuerim . . . ut ea, quae ad hanc scientiam spectant, eadem animi libertate, qua res mathematicas solemus, inquirerem sedulo curavi*, which means: "while applying myself to politics, I have sought to bring to the study of it that same freedom of mind which is usual in the study of mathematics." And he continues: "I aimed not at deriding, not at pitying, not at detesting, but at understanding human actions; at considering the human passions, such as love, hate, envy, glory, pity, not as virtues or vices, but as a property

of human nature in the same way as heat is considered to be a property of bodies."

Hence: experimental method, founded solely on facts; freedom of mind on the part of the observer, as if he were examining facts of physical nature.

Thus the philosopher of the seventeenth century and the sociologist of the nineteenth agree in the double assertion that social science is to be treated according to an experimental method, based only on facts; and that, in the presence of these facts, the observer must preserve the same freedom of mind which the naturalist preserves while investigating the life of the ants and of the bees. An absurd position, because the facts, upon which that science (sociology) raises its theoretical structures, are internal (subjective) and not external (objective): they are those that history transmits and for this reason they are *mental formations* (concepts), which thought, that is, philosophy, produces, by interpreting, considering, and qualifying actions and events. Such an interpretation is always subjective and it always reflects—consciously or unconsciously—an ethics and a mystics.

And, in contrast with the methodological assertions on which Pareto's Sociology is said to rest, it really contains *an ethics and a mystics*. The greatness of the work, from the political aspect, is precisely in this transcendental character, by which the critique of democratic society ceases to be an abstract refutation of theoretical principles, as it had been in the *Systèmes Socialistes*, and becomes a real opposition of one faith to another. Concluding his theory of economic and social cycles —several years before the world war—Pareto had foreseen that the end of the plutocratic demagogic cycle was close at hand, and in the transformation of democracy he saluted the dawn of the new day. It is one of the ironies of life that Vilfredo Pareto, the denier of every creed, of every philosophy, is the artificer who, first and most valiantly, raises—on the ruins of the democratic dogma—the edifice of the new faith and of the new philosophy, anti-democratic, anti-humanitarian, anti-progressive, anti-evolutionary. For, by taking as a foundation the critique of the derivations, that is, of the logical and pseudological reasonings with which the socialist-democratic city justified its deterministic, laical, and international faith, he restored a theoretical value to religious and patriotic values, to the principles of individual responsibility and of the freedom of the will; the principles which the wisdom of the ages has taken as the foundation of all civil life. Such a position is the more worthy of note in that Pareto's education had been conducted on rationalistic and democratic bases, when rationalism and democracy, I should say more properly the rationalistic and democratic spirit, dominated consciences undisputedly. Before a spiritual

revolution was accomplished in the multitudes, it had been accomplished in him, without any definite design of his; I would almost say, in spite of himself. A potent sign of the genius which no external force can validly resist. Just as the weakness of the flesh delayed, but could not prevent, the triumph of Saint Augustine's vocation, so a rationalistic education retarded but did not impede the flowering of the mysticism of Pareto.

For that reason, Fascism, having become victorious, extolled him in life, and glorifies his memory, like that of a confessor of its faith.

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ON THE THEORY OF BUSINESS-CYCLE CONTROL

By J. TINBERGEN

I. INTRODUCTION

THE PRESENT STUDY consists of two kinds of approach: (1) a statistical research into the chief dynamic equations describing American business cycles between 1920 and 1932 and (2) a number of more general methodological remarks on this type of research with particular attention to problems of policy. In order to simplify exposition in many points, it seemed desirable to exemplify the general thought immediately; and for this reason the two approaches are more or less mixed.

II. A MODEL OF AMERICAN ECONOMY FOR 1919–1932

In order to fix the ideas and to have a possibility of application and exemplification, a very simple model of American business life may be introduced. The variables are:

Z = Total nonlabour income earned,
F' = Expenditures by nonworkers for consumers' goods,
B = Employment in investment industries.

Z and F' are measured in such units as to make the average value of Z over the period 1919–1932 equal to one; the units of B are such that the average value of total employment over 1919–1932 equals one. The unit of time is taken equal to 4 months.

The choice of the variables and their units is less arbitrary than would seem at first sight. The model is generated, by small simplifications, from a more complicated one that would seem more logical. This has been described elsewhere.[1] The figures used for the variables and their computation are indicated in Table 1.

The equations are of the following type:

1. An institutional equation indicating the calculation of nonlabour income:

$$(1) \qquad Z_t = F_t' + B_t + \zeta_B B_{t-1} + A^0 + A^1 t + \zeta_Z (Z_{t-1} - Z_{t-2});$$

2. A reaction equation, representing the demand for investment activity:

$$(2) \qquad B_t = B^0 + B^1 t + \beta Z_t + \beta_1 Z_{t-1};$$

3. A reaction equation, indicating the spending of nonlabour income:

$$(3) \qquad F_t' = \epsilon Z_t + \epsilon_1 Z_{t-1} + C^0 + C^1 t.$$

[1] In a monograph to be published by Messrs. Hermann and Cie (Paris). Some of the figures have been improved since. For typographical reasons the notation has been changed. Capital letters in this paper correspond to double-barred variables in my other publications.

TABLE 1

VARIABLES USED IN THE SIMPLIFIED MODEL FOR THE UNITED STATES, 1919–1932

Variable (short description)	Symbol	Unit	1919	1920	1921	1922	1923	1924	1925	1926	1927	1928	1929	1930	1931	1932
Nonlabour income earned (including business savings)	Z_t**	Average, 1919–32, =100	107	98	70	89	101	105	124	119	122	138	132	90	62	42
Work done in investment industries	B_t**	[a]	23	25	16	20	24	21	22	24	21	21	23	18	13	10
Nonworkers' consumption outlay	*	[b]	80	71	68	76	72	77	83	92	90	94	98	81	70	68

[a] Unit such as to make total work done in all industries equal to 100 in the average for 1919–1932.

[b] With my figures this unit makes the average value of Z over 1919–1932 equal to 100; therefore the unit for Z is also $\$34.4 \times 10^9$ and equals that for F.

Computation of these values.

Z_t Based upon data given in *America's Capacity to Consume* (Moulton, Leven, Warburton, et al.) for the years up to 1929 and upon the official estimates of the Department of Commerce (Kuznets). The figures include not only dividends, interest, net royalties and rents, and entrepreneurial withdrawals, but also one-half of the salaries and, finally, estimates for speculation gains, as given in the above work.

B_t Employment figures for the metal industries multiplied by such a factor as to make the average of total employment in industry equal to 100 for the period 1919–1932. It is assumed that the average level of employment in investment industries covers the same proportion of total employment as the proportion between gross capital formation (taken from Kuznets, *Bulletin of the National Bureau of Economic Research*, No. 52) and national income.

F_t Found by subtracting the total wages bill (i.e., in the terminology of the above-cited sources on national income, wages plus one-half of the salaries) from the total value of consumers' goods. The latter item has been taken for 1919, 1921, 1923, 1925, 1927, and 1929 from Warburton's estimates ("Value of the Gross National Product and its Components, 1919–1929," *Journal of the American Statistical Association*, Vol. 29, Dec., 1934, pp. 383–388), and for the other years found by interpolation and extrapolation with the aid of F.R.B. indices, agricultural, automobile, mineral, and railway statistics.

It should be borne in mind that the relevant features in our figures are only their movements around their trends and that these movements are only slightly changed if some alternative methods of computation are used.

The justification for this special choice of the form of our equations has been given elsewhere.[2]

The quantities A^0, A^1, B^0, B^1, C^0, C^1, ζ_B, ζ_Z, β, and ϵ are supposed to be changing only slowly and are treated as constants.

III. STATISTICAL DETERMINATION OF EQUATIONS

Two methods are possible, in principle at least, which may be called the *structural* and the *historical* method.

The structural method tries to measure immediately the constants appearing in the equations or to deduce them by reasoning in connection with direct measurement. In order to do so, one must be clear about the meaning of the coefficients. An example will be given while discussing the American model in this respect.

The historical method is that usually known as the method of correlation analysis. The chief difficulties are:

1. Theoretical:

(a) The possibility that disturbances do not follow a simple statistical law of distribution;

(b) The possibility of multicollinearity;

2. Practical:

(a) The availability of no data or only bad data;

(b) The availability of only annual figures, making difficult the exact determination of lags.[3]

The problems met in calculating the coefficients of equations (1) to (3) are in many respects illustrative of these difficulties. Some details may be given.

Equation (1): This equation contains, strictly speaking, not only the coefficients A^0, A^1, ζ_B, and ζ_Z as unknown coefficients, but also the time structure of the corresponding terms, i.e., the lag of the term $\zeta_B B_{t-1}$ and the two lags included in the term $\zeta_Z(Z_{t-1}-Z_{t-2})$. Now it is scarcely possible to determine these time structures really exactly with the help of only annual data, which we may indicate by

$$Z_t^{**} = Z_{t-1} + Z_t + Z_{t+1} \text{ for the year centered at time } t,$$

and

$$Z_{t-3}^{**} = Z_{t-4} + Z_{t-3} + Z_{t-2} \text{ for the preceding one, etc.}$$

For that reason they have been assumed a priori to be as they are written down. The considerations upon which the choice is based are:

[2] Cf. (a) *Revue Trim. de l'Institut Int. de Stat.*, July, 1936; (b) *Weltw. Archiv*, Vol. 45, 1937, p. 42; (c) *Archiv für math. Wirtschafts- und Sozialforschung*, Vol. 3, 1937, p. 7.

[3] In fact only annual figures are available for the two variables Z and F' which are of vital importance to our analysis.

I. The meaning of the lag in $\zeta_B B_{t-1}$ is that it equals about one-half of the period of production in investment-goods industries; a period of production of 8 months seems to be a fair assumption, and therefore $\zeta_B B_{t-1}$ was supposed to lag 4 months. Furthermore, B_{t-1} has been approached by the formula

$$(I) \qquad 3B_{t-1} = 2B_1 + B_{t-3} = \tfrac{2}{3}B_t^{**} + \tfrac{1}{3}B_{t-3}^{**},$$

which holds exactly only in the case of rectilinear development of B, but approximately also for the cases in which we are interested. The right-hand member of (I) has, therefore, been introduced as one variable in our calculations.

II. Statistically speaking, the only practical possibility to represent speculative gains with the help of only annual figures is to take consecutive differences. In our notation, this would be $\zeta_Z(Z_t^{**} - Z_{t-3}^{**}) = \zeta_Z[(Z_{t+1}+Z_t+Z_{t-1}) - (Z_{t-2}+Z_{t-3}+Z_{t-4})]$. Now if Z shows a movement that for intervals of two years may be approximated by a straight line, this expression may be simplified in various ways. In our example we choose the one leading to the simplest formulae, viz.:

$$(II) \qquad \zeta_Z(Z_t^{**} + Z_{t-3}^{**}) = 9\zeta_Z(Z_{t-1} - Z_{t-2}).$$

As the hypothesis of a rectilinear development of Z_t is only a very rough approximation, it would be interesting to make a refined investigation with perhaps interpolated values for Z. As my purpose is, however, only to give an example, I do not go into this question now.[4]

With this given time structure, the coefficients A^0, A^1, ζ_B, and ζ_Z have been determined by the historical method applied to the period 1920–1931.[5] Using the elementary regression on Z_t^{**} one finds:

$$Z_t^{**} = (F_t' + B_t)^{**} + 2.1B_t^{**} + 1.05B_{t-3}^{**} + 0.36(Z_t^{**} - Z_{t-3}^{**}) + 0.81t^{**} - 66.$$

Here, as is indicated by the double asterisks, all figures are annual totals, as given in the tables. The symbol t^{**} indicates time in half years elapsed since the middle of the period taken (i.e., Dec. 31, 1926). Writing for the double asterisk symbols their expressions and using (I) and (II), the above equation can be transformed into one for "four-month variables." At the same time t^{**} should be replaced by t (whose units are 4 months).

We obtain:

$$(4) \qquad Z_t = F_t' + B_t + 3.15B_{t-1} + 1.08(Z_{t-1} - Z_{t-2}) - 22 + 0.18t.$$

[4] Also in some cases of nonrectilinear development the approximation chosen would be valid.

[5] The year 1932 has not been used, because afterwards the impression was obtained that a strong, more or less extra-economic, shock was produced by the fall of sterling.

The coefficient ζ_B may be checked by the structural method. A closer analysis of the meaning of the term $\zeta_B B_{t-1}$ tells us that it is generated from the expression, "value of means of production produced minus wages paid in corresponding industries," from which it differs by a con-

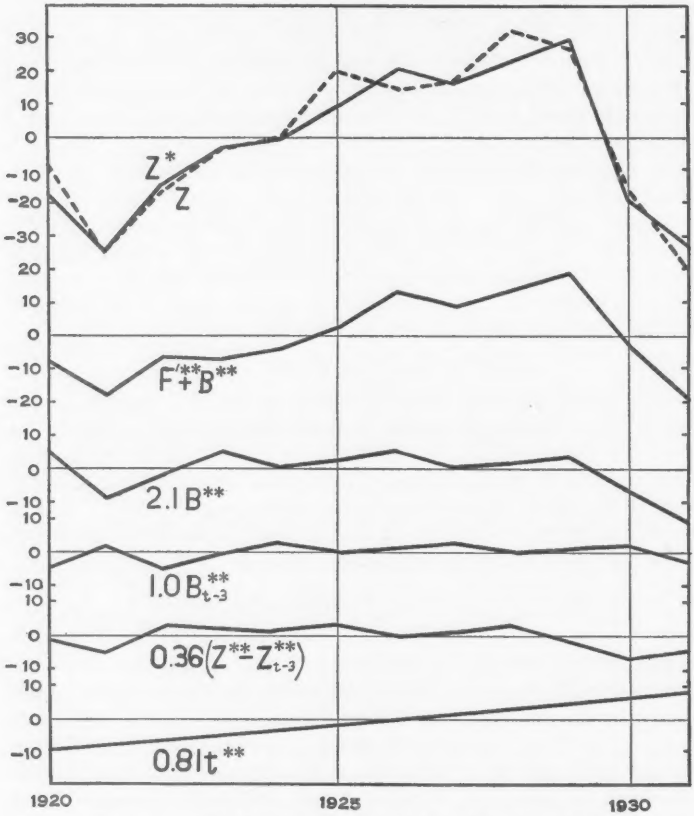

FIGURE 1.—Explanation of movements in Z by formula (1).

stant plus a trend. Writing Q_t for the prices of means of production and Y_t for their production, then the part of Z_t corresponding to investment industries should be:

$$Q_{t-1}Y_{t-1} - L_{t-1}B_{t-1}.$$

In this expression, Q and Y move about parallel to B; the relation between Q and B depending on the changes in productivity, and that

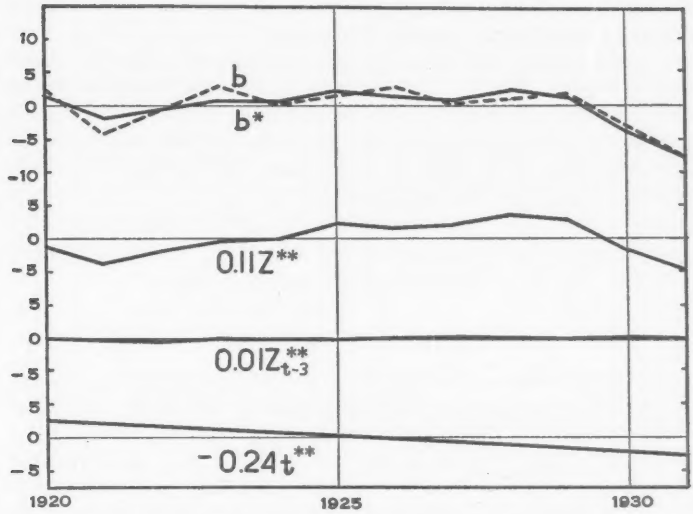

FIGURE 2.—Explanation of movements in b by Formula (2).

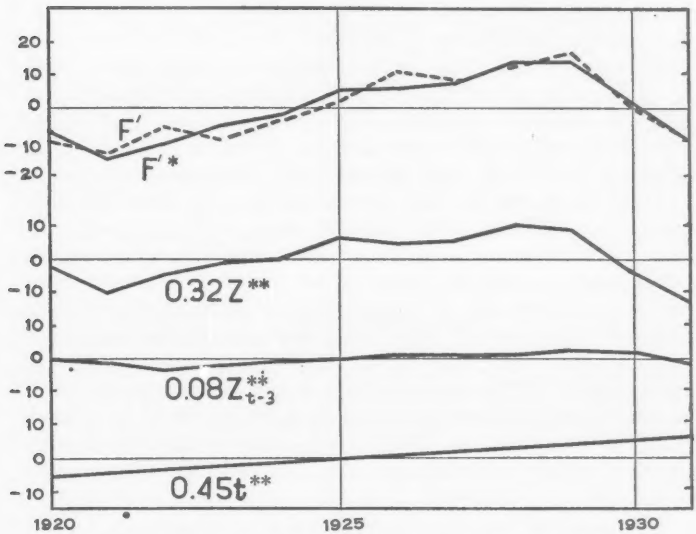

FIGURE 3.—Explanation of movements in F' by formula (3).

between Y and B depending, in addition, on the slope of the supply curve. A rough check shows that the coefficient found is realistic.

Equations 2 and 3: To these equations the historical method has been applied in such a form that a regression equation has been established with the values for Z in the same year and the year before and with time as independent variables. This equation has been reduced to an equation in Z_t and Z_{t-1} (i.e., the values for t and for 4 months before) with the help of the relation, similar to (I):

$$3Z_{t-1} = \tfrac{2}{3}Z_t^{**} + \tfrac{1}{3}Z_{t-3}^{**} \tag{III}$$

in the notation just introduced. Using elementary regressions, we find

$$B_t = 0.09Z_t + 0.03Z_{t-1} + 2.3 - 0.05t \tag{5}$$

and

$$F_t' = 0.16Z_t + 0.24Z_{t-1} + 0.10t. \tag{6}$$

Figures 1, 2, and 3 show for each of the three variables: (i) the observed values (deviations from average), dotted line; (ii) the calculated values (with asterisk), first full line; and (iii) the components of the calculated values. The latter are indicated by their mathematical expressions, which are given in terms of the annual figures used in the correlation analysis and given in the table. For that reason the coefficients are not those of (4), (5), and (6); these latter would be obtained by applying (I), (II), and (III).

IV. EQUILIBRIA AND EQUILIBRIUM DEVELOPMENTS

A first step to find the properties of a given economic system, represented by its system of dynamic equations, is the determination of its equilibria and its equilibrium developments. By an equilibrium I understand a set of time-constant values of variables satisfying the equations. If such a set exists, the economic system possesses a stationary state. In general this will not be the case, but there will exist, among the possible movements of the system, movements that are distinguished by a more continuous, a more smooth development than other possible movements. I have called them "equilibrium developments" and shown that an economic system represented by a linear system of equations with only linear functions of time as "known terms" is able to carry out a linear movement in time. Systems with more complicated "known terms" will, in general, show more complicated "simplest movements," which, in their case, could be taken as equilibrium developments. But anyhow such an equilibrium development can be found[6] and, in the case of linear equations with arbitrary

[6] Cf. my paper, "Einige Grundfragen der mathematischen Konjunkturtheorie," *Archiv für mathematische Wirtschafts- und Sozialforschung*, Vol. 3, 1937, p. 1.

"known terms," it is always possible to obtain homogeneous equations (i.e., equations without "known terms"). This can be done by measuring all variables by their *deviations from the equilibrium development* instead of by their absolute ("natural") values. These deviations would be comparable to trend deviations in the statistical language. The movements of these deviations, which are in fact the movements in which business-cycle research is most interested, would thus be governed by simpler equations easier to handle mathematically.

In our example this would lead us to the following set of equations for the deviations from the equilibrium development:

$$z_t = f_t' + b_t + 3.15b_{t-1} + 1.08(z_{t-1} - z_{t-2}), \tag{7}$$

$$b_t = 0.09z_t + 0.03z_{t-1}, \tag{8}$$

$$f_t' = 0.16z_t + 0.24z_{t-1}. \tag{9}$$

In these equations the lower-case symbols are the deviations corresponding to the capital ones used previously.

The equilibrium position or development may be *stable* or *unstable;* this depends on the nature of the small movements generated by some small initial deviation from equilibrium. Stable equilibrium will be present if these movements tend to approach, in the long run, the equilibrium.

V. NECESSITY OF SIMPLIFICATIONS

In at least three respects the business-cycle mechanism is complicated: (1) as to the economic structure (the number of variables and relations is large); (2) as to the time structure (the number of different lags and lag distributions is large); (3) as to the mathematical structure (complicated functions play a role). In order to be workable at all, the model must be simplified. Many interesting features of business-cycle problems can only be seen after the solution of the equations; and in order to be solved these must be workable.

Simplifications in economic structure. Two types of simplifications may be mentioned in this section, viz., "macro-analysis" and "incidental simplifications." By "macro-analysis" (Frisch) is meant the grouping of a number of variables of the same type as index numbers or totals (price indices, value totals, etc.). It is common to almost all business-cycle theories and could scarcely be dispensed with. How far it is legitimate is, however, not easy to see. This will, nevertheless, not be discussed here; it seems to me that only concrete figures can lead here to fruitful statements.

"Incidental simplifications" are present as soon as in a certain equation some terms are neglected; this will in general be justified by the smallness of these terms. A general case is the neglecting of all influences that are almost constant or trend influences. For this reason, the

demand for replacement of capital goods may be neglected, as far as its deviations from trend are concerned.

Another example of an important "incidental simplification" is the neglecting of the influence of interest rates (in the narrower sense of short or long market rates for lending) on investment, which, it seems to me, is in many cases justified.[7]

An interesting type of simplification has been suggested by Mr. Haavelmo at the Oxford meeting of the Econometric Society, 1936. It might be formulated as follows. Suppose we have a system with n variables. Suppose further that we are only interested in the movements of a small number m of these variables, relating to one special field or market and that it is possible to eliminate a number of the variables in which we are not interested. We may then be left with one or more equations between the noninteresting variables, which no longer represent direct causal relationships but only indirect ones, as they are elimination results. The coefficients of such equations and even their mathematical form may then be obtained immediately from observation, instead of being found by elimination calculations with elementary coefficients derived themselves from observation. In other words, these equations may immediately be deduced from observations. They are, however, no longer interpretable; they are perhaps numerically more exact and easier to handle in further computations.

VI. SIMPLIFICATIONS IN TIME STRUCTURE

Simplifications in time structure may be, above all, simplifications in the lags occurring in the system. In principle, a considerable number of the most different lags will exist, which are even incommensurable. An important step towards simpler formulas is their replacement by commensurable ones. In addition, their number may be reduced by neglecting small ones or by reducing "distributed lags" (Irving Fisher) to simpler ones. An example of the latter will be given.

It will be noted that our illustrative system shows a very much simplified time structure. I think this is in so far legitimate, as static theory did not consider any time structure at all, and we have, therefore, still to learn the very simplest consequences of the introduction of time structures. These very much simplified time structures may, however, lead into oversimplifications where even the relevant features are eliminated. A clear example can be quoted. Suppose that we simplified our model to such an extent as to be left with only two different time points in the final equation. This could always be attempted by taking as the time unit the largest lag appearing in that equation and by replacing a lag of intermediate length by a combination

[7] I hope to have an opportunity to go into this problem more at length.

of a zero lag and a unit lag. *In concreto:* if two years were the largest lag appearing in a final equation

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} = 0 \quad \text{(time unit 1 year)},$$

we could write for $a_1 x_{t-2}$: $\frac{1}{2}a_1 x_t + \frac{1}{2}a_1 x_{t-2}$, which under certain conditions is not so bad an approximation, and which would change our final equation into:

$$(1 + \tfrac{1}{2}a_1)x_t + (a_2 + \tfrac{1}{2}a_1)x_{t-2} = 0 \text{ (time unit 1 year)},$$

or

$$(1 + \tfrac{1}{2}a_1)x_t + (a_2 + \tfrac{1}{2}a_1)x_{t-1} = 0 \text{ (time unit 2 years)}.$$

This would, however, be simplification of a dangerous nature, as such an equation can no longer show cycles of any length, but only very short ones, or no waves at all.[8]

Of course, every reduction in the number of terms of the final equation means a certain "impoverishment" of the picture. A four-term equation may show two superposed cycles, whereas a three-term one can only show one cyclic component, etc. But it seems that, in particular, the reduction of a three-term equation to a two-term one is a heavy impoverishment, as the movements corresponding to a three-term equation still may show any period, whereas those of a two-term one can only be either very short-period or nonperiodic movements.*

VII. SIMPLIFICATIONS IN MATHEMATICAL FORM OF RELATIONSHIPS

A third type of simplifications is obtained by the introduction of simpler types of functions instead of the functions that in principle should have been chosen. One very general procedure is the approximation by linear functions of all relations entering into the system. This is a powerful help for further calculations, as linear equations are in many cases solvable explicitly, which is practically never so for nonlinear equations.[9]

For this reason I prefer linear formulas for demand functions, etc.

[8] Even in the latter case, however, the first extrapolations obtained by such an equation may have a certain value. Cf. Section XI below.

* Even such a system, whose characteristic equation gives only one component, namely an exponential (apart from trivial cycles shorter than the distance between observations), will, however, produce real observable cycles when it is exposed to erratic shocks. These cycles will be "changing harmonics" and their average frequencies can be determined from the coefficients of the characteristic equation. Investigations which Mr. Haavelmo and I have quite recently undertaken on American data have led us to suspect that many structures in economic reality have, from the viewpoint of the characteristic equation, only an exponential component.—EDITOR'S FOOTNOTE.

[9] A closer examination of the implications of such a type of simplification was considered in my paper: "Über den Wert mathematischer Konjunkturtheorien," *Beiträge zur Konjunkturlehre*, Hamburg, 1936.

instead of constant-elasticity formulas in cases where the statistical material is not strongly against this hypothesis. When the linear hypothesis is adopted it is not important whether the elasticity of a certain demand or supply function is different in boom and depression periods, but it is important whether its slope is different.[10]

The linear approach is, however, too rough in some cases of special importance to business-cycle theory. A good example is that of the phenomenon of saturation, which, for example, may show itself in the following case: Suppose full employment is reached without a turning point occurring. The boom may then still continue and show itself in higher wages, prices, etc. Evidently, however, the relationship determining the volume of employment must show curvilinearity in some respect. To take the case of our equation (8), b_t cannot continue to be the same linear function of z_{t-1}, after the state of full employment, say b_f, has been reached. It will, in fact, be a relation showing an asymptote for $b_t = b_f$, and an extrapolation of our equation (2) for $b_t > b_f$ will not be admissible. As, however, it would be rather difficult to work with non-linear equations, one might here use the method of approximating the asymptotic function by two functions. Of these, the first would be equation (8), valid only for $b_t < b_f$, whereas the second would be $b_t = b_f$, valid only if $z_{t-1} \geqq b_f/\beta$. Probably other equations would have to be changed at the same time, e.g., the equation determining q_t; because one has to expect that the impossibility of a further increase in production, aside from a still growing demand for investment goods, will raise prices, although b_t remains constant.

The discussion of the system's movements must then be split up into two parts; first, a discussion for $b_t < b_f$, and secondly, one for $b_t = b_f$. The systems of equations would be, e.g.:

Case I: $b_t < b_f$	Case II: $b_t = b_f$
$z = f' + 2b_{-1} + 0.4q + \zeta z(z_{-1} - z_{-2})$	$z = f' + 2b_{-1} + 0.4q + \zeta z(z_1 - z_{-2})$
$b = \beta z_{-1}$	$b = b_f$
$f' = \epsilon_1 z + \epsilon_2 z_{-1}$	$f' = \epsilon_1 z + \epsilon_2 z_{-1}$
$q = kb_{-1}$	$q = k' z_{-1}$

The first and third equations would have remained the same; but the second and fourth have changed. Whether the fourth equation would become $q = k' z_{-1}$ or perhaps $q = k' z_{-2}$, or still another one, would have to be seen from observations. The chief remark to be made now is that the character of the movements may have been changed. It might be, e.g., that in Case II a shorter period is shown than in Case I. It might even be that Case I does not show a periodic movement at

[10] Haberler, *Depression and Prosperity*, Geneva, 1937, attaches a great importance to changes in elasticities for the explanation of the "turning points."

all, whereas Case II might do so. In that case, Haberler's distinction between the analysis of the cumulative movement and the turning-point analysis would be translated in a simple way into mathematical formulae.[11] The two cases would at the same time correspond to the "first" and the "second" stage of the boom ("erster und zweiter Anstieg" of Spiethoff). It must be added that for the United States between 1919 and 1932 a state of full employment in the investment industries with a corresponding overnormal rise in prices has hardly existed and that in that case our former analysis seems to apply. The history of German business cycles before the war seems, however, to show interesting examples of the second stage of the boom.

There are still other forms of simplifications. They were already indicated in my "Survey" and the reader may be referred to that paper.[12]

VIII. CHARACTER OF MOVEMENTS

The character of the system's movements can be determined most easily and explicitly in the following special case: (a) it is possible to find one "final equation" by an elimination of all variables but one and this equation is linear. It is easy to see then that two sorts of components are, in general, possible: cyclical and noncyclical; the latter either tending to an equilibrium[13] or moving away from an equilibrium. Although this case is a special one, many more general cases can, with a certain degree of approximation, be reduced to it.

In the case of a nonlinear final equation there exist other equilibrium positions or developments than the one chosen as reference development and much depends on the situation of these other equilibria, which may be stable or unstable. These can be found by supposing the deviations from trend of the variable left as constants and solving the final equation. If there are several such equilibrium situations within the field of economic relevant values, the movements may be far more complicated.[14]

In our simple example we can easily obtain one final equation by eliminating b_t and f_t'; we are left with an equation in z_t:

$$(10) \qquad \begin{aligned} 0.75z_t - 1.63z_{t-1} + 0.99z_{t-2} &= 0 \text{ or} \\ z_t - 2.18z_{t-1} + 1.32z_{t-2} &= 0. \end{aligned}$$

[11] Haberler, *op. cit.*

[12] "Annual Survey: Suggestions on Quantitative Business Cycle Theory," ECONOMETRICA, Vol. 3, July, 1935, pp. 241–308.

[13] This term being understood now to include also "equilibrium developments" (cf. section IV).

[14] I treated (constructed) examples in the following papers: "Über den Wert mathematischer Konjunkturtheorien," *loc. cit.*, and "Über die Sekundarwirkungen zusätzlicher Investitionen," *Weltwirtsch. Archiv*, January, 1937, p. 39.

The characteristic equation corresponding to this difference equation is

$$x^2 - 2.18x + 1.32 = 0,$$

with the roots

$$x_1 = 1.09 + 0.36i \quad (i = \sqrt{-1}),$$
$$x_2 = 1.09 - 0.36i.$$

The general solution of (10) is, therefore,

$$z_t = c\sqrt{1.32}\, t \sin (19.6t + d), \tag{11}$$

representing a cyclic movement with a period of some 6 to 7 years, with rather rapidly increasing amplitude.

IX. VARIATION PROBLEMS

These are the problems of the influence of a change in constants on the movements. The setting of these problems is in itself often difficult. Formulated rather generally, the difficulty is that a change in one constant often entails a change in another and that to assume only the first constant changed would be economic nonsense. A circumstance of special importance belonging to this type of difficulty is that the movements of a system are not determined by the coefficients of the equations only but also by the initial values of the variables. Therefore it must be known whether an assumed change in some coefficient entails a change in initial values. Examples may illustrate the difficulty.

Compensating public works. The shifting of public works from prosperity to depression may be considered as a case of changing coefficients, viz., reducing the β's, say, to one-third of their former values. Instead of our final equation (10) we now get

$$0.81z - 1.42z_{-1} + 1.05z_{-2} = 0$$

or

$$z - 1.75z_{-1} + 1.30z_{-2} = 0.$$

The roots of the characteristic equation

$$x^2 - 1.75x + 1.30 = 0$$

are

$$x_{1,2} = 0.87 \pm 0.74i,$$

showing that the movements are slightly less explosive but have a considerably shorter period, viz., of about 3 years.

A question of large practical importance, viz., whether the amplitude of the cycles will be diminished by such a policy cannot, however, be answered yet by this statement. Amplitudes are largely determined by initial conditions; only relative amplitudes are fixed by the co-

efficients of the equations. Thus it is quite clear that the proportion of the amplitude of b to that of z is diminished; but the question remains whether the amplitude of z will be larger or smaller. This apparently depends on whether the initial shock leading to the cyclical movements usually appears in b, in z, in both, or in other variables. A multitude of possibilities presents itself.

(i) Suppose, first, that a system usually is brought out of equilibrium by only one shock in only one variable and that in this way the cycle is generated. If, now, this shock is caused by an important invention, the first deviation from equilibrium will show itself in b, representing investments for new industries, not based on previous profits. This initial deviation in b will then be the same in the case without and in that with compensating public works; and the corresponding initial deviation in z will also be the same, as it is found from equation (1). The further development will be different because of a difference in period as well as one in dampening degree. As the period is shorter in the case with compensating public works, the system cannot reach such a large amplitude as it could without compensating public works. As, in addition, the dampening degree is somewhat smaller, this conclusion is reinforced. The influence of the policy discussed will, therefore, in this case, be a decidedly favorable one, far more than one would deduce from the (only small) change in the damping degree.

(ii) Suppose now, however, that a change in crops is usually the initiating cause. In our simplified model this would mean that z at that moment shows a value different from the one found by formula (1): the crop change would change nonlabour income for another reason than those represented in that equation. The initial deviation would now occur in z and, therefore, be the same in the case without and in that with compensating public works. Because of equation (2) we should now find a different initial deviation in b in the two cases. It would be smaller in the case with compensating public works. As, in addition, the same arguments of period and dampening changes apply, the amplitude of b will now be reduced still more than formerly by the compensating policy.

(iii) In general, however, a complex of causes, partly from remote times, will be the origin of disequilibrium; and then the problem is somewhat more complicated. In every case, however, the *"initial values" influence the result of any form of business-cycle policy*. And these initial values are not only an expression for the phase of the cycle in which society finds itself at the moment the policy starts, but also for the "localization" of new disequilibria likely to occur: for the fact whether these disequilibria are more concentrated upon one part or more upon another part of society.

X. FURTHER EXAMPLES ON BUSINESS-CYCLE POLICY

We continue our examples of variation problems by considering the consequences of a reduction in speculation opportunities, leading, say, to a reduction of the realized speculative gains to one-half of their previous value. This would mean, in our mathematical language, that the equation for z is changed into:

$$z = f' + b + 3.15b_{-1} + 0.54(z_{-1} - z_{-2}).$$

The other equations would not have changed. In this case the final equation becomes:

$$z - 1.45z_{-1} + 0.60z_{-2} = 0.$$

The roots of the characteristic equation are:

$$x = -0.72 \pm \sqrt{0.52 - 0.60} = -0.72 \pm 0.28i;$$

from which we find that the period of the movements is now $5\frac{2}{3}$ years. And—which is far more important—the movement has become a very much damped one; the amplitude, after about half a period, is reduced to less than 0.13. The business cycle will, therefore, to a large degree be eliminated by this measure.

Next consider a *change in savings habits*. Suppose there is more saved in the boom and less in depression, or, which is the same thing, that expenditures are less fluctuating. This would mean, in our language, that the coefficients of the equation for f' would become smaller, say $\frac{1}{2}$ of their former value. We get, then, the final equation,

$$0.83z - 1.51z_{-1} + 0.99z_{-2} = 0.$$

This corresponds to a movement with 3.6-year cycles and with a somewhat more stable amplitude (a smaller "explosiveness"). The improvement as compared with the uninfluenced case is not very large. A definitely unfavorable change would, however, take place if savings habits were changing in the opposite direction, viz., if savings were more stable throughout the cycle.

Of course, all that has been said in Section IX about the influence of initial values still applies.

XI. REMARKS AND FIGURES ON THE SIGNIFICANCE OF RESULTS

I finish this paper with some remarks on the "margin of arbitrariness" in the results.

Beginning with the restricted question of significance in the usual statistical sense, I need not repeat here that there is still much difference of opinion as to what is the best criterion by which to judge the limits of uncertainty in regression coefficients. For instance, instead of the elementary regressions given above, others may be used. A type of

mean regression I often make use of is that obtained by dividing the elementary regression coefficients by the total correlation coefficient. In the case of two variables only, this type is identical with the diagonal regression. The equations obtained by this method may be caledl "Set 2" against the original "Set 1." Set 2 then runs:

$$z = f' + b + 3.3b_{-1} + 1.14(z_{-1} - z_{-2}),$$
$$b = 0.10z + 0.03z_{-1},$$
$$f' = 0.167z + 0.252z_{-1},$$

leading to the final equation:

$$z - 2.40z_{-1} + 1.42z_{-2} = 0.$$

There are, however, a number of other arbitrary assumptions at the basis of our equations.

Consider, for instance, the assumption made as to the lag between z and b (cf. Section III, Subsection I). I shall replace this assumption by another, viz., by one that could be called "the introduction of a free lag." The procedure consists in including in the correlation calculation for z both b_t** and b_{t-3}** as independent variables and transforming the result in the usual way (cf. Section III) in order to obtain an equation with b_t and b_{t-1}. This procedure again can be carried out by using an elementary regression equation or by using the sort of mean regression equation just mentioned. The following two alternatives are obtained:

$$z = f' + 1.9b_{-1} + 3.0b_{-2} + 1.47(z_{-1} - z_{-2}),$$
$$z = f' + 1.9b_{-1} + 3.1b_{-2} + 1.50(z_{-1} - z_{-2}),$$

to be used, respectively, in "Set 3" and "Set 4."

All the calculations about the movements of the system and the influence of given types of business-cycle policy can now be made with the four sets.

(i) Starting with the calculation of the period of the undisturbed system we find that "Set 2" shows a noncyclic movement, the roots of the characteristic equation being

$$x = 1.20 \pm 0.14.$$

For our purpose—discussion of policy—this is not so serious. What matters is extrapolation for the first few time units (of four months).

(ii) An extrapolation for the first two or three time units gives quite analogous results for the four sets, if we start with a "shock" in a given direction, e.g., $z_{-2}=0$; $z_{-1}=+1$. We find:

Set 1:	$z_0=2.18$	$z_1=3.43$	$z_2=4.5$
Set 2:	2.40	4.34	7.0
Set 3:	2.13	3.39	4.8
Set 4:	2.21	3.75	5.8

All sets show a "self-reinforcing, cumulative process." The same would, of course, be the case, had a downward "shock" starting from $Z_{-2}=0$ been chosen.

(iii) The results are, however, less homogeneous if we start with an almost horizontal movement, e.g., $z_{-2}=1$, $z_{-1}=1.1$. We find:

Set 1:	$z_0=1.08$	$z_1=0.90$
Set 2:	1.22	1.37
Set 3:	1.20	1.31
Set 4:	1.31	1.67

Again, the divergencies in the levels are not very large, which fact for practical uses may still be of importance.[15]

For the three problems of business-cycle policy we considered above, the following results are found:

Compensating public works

1. $x^2-1.75x+1.30=0$	Period: 36 months
1. $x^2-1.89x+1.39=0$	39 m.
3. $x^2-2.07x+1.55=0$	43 m.
4. $x^2-2.14x+1.59=0$	45 m.

Restriction of speculation opportunities

1. $x^2-1.45x+0.60=0$	Period: 68 months
2. $x^2-1.62x+0.64=0$	
3. $x^2-1.25x+0.26=0$	48 m.
4. $x^2-1.32x+0.24=0$	

Expenditures more stable

1. $x^2-1.82x+1.19=0$	Period: 43 months
2. $x^2-1.98x+1.27=0$	51 m.
3. $x^2-1.82x+1.04=0$	53 m.
4. $x^2-1.87x+1.03=0$	62 m.

Combining the three types of policy

1. $x^2-0.85x+0.57=0$	Period: 26 months
2. $x^2-0.91x+0.61=0$	26 m.
3. $x^2-0.96x+0.61=0$	28 m.
4. $x^2-0.98x+0.62=0$	28 m.

Since, for cyclic movements, the damping per 8 months is given by the last term of the characteristic equation (in the form given above), we see that a favorable influence on damping is exercised by restric-

[15] For a fuller exposé of the question whether a separate explanation of turning points is necessary or not, cf. my article "Einige Grundfragen der mathematischen Konjunkturtheorie," *Archiv für mathematische Wirtschafts- und Sozialforschung*, Vol. 3, 1937. Heft 2.

tion of speculation and by stabilizing expenditures, in so far as they lead to cyclic movements. Set 2 and Set 4, however, do not show a cyclic movement in the case of speculation restriction. Therefore,other criteria must be used for those cases. These are in the roots of the characteristic equation. Both roots have a modulus below 1 for Set 2, which also means a damped movement. Only in Set 4 one of the roots is 1.1, indicating the possibility of moderately "explosive" movements; these are, however, still less explosive than the movements possible in the uninfluenced system corresponding to Set 2.

From these and similar analyses we see that the results of various types of policy *are less uncertain than the movements of the uninfluenced systems*. The smallest spread is shown, however, just in the case most important in practice, viz., the case of a combined policy in the three directions chosen. In fact, the combination shows a very favorable result for all four sets: strongly damped movements with short periods, not differing very much among themselves.

Thus our conclusions concerning (1) the explanation of a cumulative, self-reinforcing process and (2) the effects of business-cycle policy seem more certain than those concerning (1) the explanation of the turning points and (2) the length of the period. This is a rather happy situation, in so far as in fact the chief purpose of business-cycle policy is to prevent the development of cumulative processes.

More generally, if all our equations are of the shape:

$$\begin{aligned} x_t &= \alpha_{11}x_{t-1} + \alpha_{12}x_{t-2} + \cdots + \beta_{11}y_{t-1} + \beta_{12}y_{t-2} + \cdots \\ &\quad + \gamma_{11}z_{t-1} + \gamma_{12}z_{t-2} + \cdots, \\ y_t &= \alpha_{21}x_{t-1} + \alpha_{22}y_{t-2} + \cdots + \beta_{21}y_{t-1} + \beta_{22}y_{t-2} + \cdots \\ &\quad + \gamma_{21}z_{t-1} + \gamma_{22}z_{t-2} + \cdots, \\ z_t &= \alpha_{31}x_{t-1} + \alpha_{32}x_{t-2} + \cdots + \beta_{31}y_{t-1} + \beta_{32}y_{t-2} + \cdots \\ &\quad \gamma_{31}z_{t-1} + \gamma_{32}z_{t-2} + \cdots, \end{aligned}$$

it can easily be seen that a *reduction of all coefficients* in the right-hand members, if only effectuated at a sufficiently large degree, finally must lead to a suppression of any disturbance of equilibrium.

League of Nations
Geneva

PURE ECONOMICS AS A STOCHASTICAL THEORY

By K.-G. HAGSTROEM

THE CLASSICAL THEORY

CONSIDER TWO commodities, (0) and (1), the first being denoted as *money*, and the other being for the sake of representation identified with a certain kind of bills, say payable 20 years hence. By a possible contract (x, y), we shall mean that A buys the amount x of bills from B, paying for it in money y units, the "price" being $p = y/x$. In the special case which we shall use as an illustration, we may put

$$y = \frac{x}{(1 + i)^{20}}$$

i denoting a rate of interest according to which the price is determined.

The effect upon the fortune implied by the signing of the contract (x, y) may be thus stated:

$$\Delta = \pm \left(\frac{x}{(1 + i)^{20}} - y \right)$$

the sign + referring to the buyer, who increases his fortune by x units of (1) and diminishes it by y units of money (0), and the sign − to the seller.

The analytical apparatus of pure economics, as illustrated, e.g., by Bowley's[1] discussion of the simple exchange, ought to be applicable to the present case.

If $\alpha(x, y)$ and $\beta(x, y)$ are the ophelimity functions for the two individuals, the "point of exchange" according to this theory is defined as the point of intersection of the "curves of exchange"

$$x \frac{\delta\alpha}{\delta x} + y \frac{\delta\alpha}{\delta y} = 0 \qquad \text{and} \qquad x \frac{\delta\beta}{\delta x} + y \frac{\delta\beta}{\delta y} = 0.$$

This lies also on the "contract curve":

$$\frac{\delta(\alpha, \beta)}{\delta(x, y)} = \begin{vmatrix} \frac{\delta\alpha}{\delta x} & \frac{\delta\alpha}{\delta y} \\ \frac{\delta\beta}{\delta x} & \frac{\delta\beta}{\delta y} \end{vmatrix} = 0.$$

NECESSITY OF A STOCHASTICAL THEORY

The *curvatures* of the curves, $\alpha(x, y) = \text{constant}$ and $\beta(x, y) = \text{constant}$, are essential in the classical theory. However, they can hardly

[1] *The Mathematical Groundwork of Economics*, Oxford, 1924.

be said to represent the most prominent feature in the economic process. The simple case of *linear* ophelimities should not be discarded as a limiting case, where the price determination fails and no real equilibrium exists. In fact, if

$$\alpha(x, y) = \alpha x - y, \quad \beta(x, y) = y - \frac{1}{\beta} x, \quad \alpha \text{ and } \beta \text{ constants},$$

we have two cases to distinguish: either the subjective price of the buyer, α, is less than that of the seller, $1/\beta$, or we have the opposite case $\alpha \geqq 1/\beta$. In the former case, there is evidently no contract (x, y) where the ophelimities of A and of B would be positive together; in the latter case, there is a domain of prices,

$$\frac{1}{\beta} < p < \alpha,$$

where both functions, $\alpha(x, y)$ and $\beta(x, y)$, are positive. All contracts (x, y), for which y/x is a number p of this domain, will be possible, and the ophelimities,

$$\alpha(x, y) = x(\alpha - p), \qquad \beta(x, y) = x\left(p - \frac{1}{\beta}\right),$$

will tend towards infinity with x, the quantity of the commodity (1) which, according to the contract, is exchanged against y units of money. Although no definite equilibrium in the usual sense of the word exists in this case, these extreme alternatives do nevertheless throw considerable light on the problem. There ought to be something in the straight-line case which it is a matter of interest to study. In the classical theory, this is not done; here it is precisely the small deviations from straight lines which determine the point of exchange, i.e., the most favourable contract.

If straight lines are considered, the viewpoint of the *calculus of probabilities* become important. We have undertaken to treat the theory from this point of view. In fact, if there is a field where this mathematical theory should be at home, it ought certainly to be the theory of the equilibrium of exchange. The deliberations which form the practical aid in every kind of business naturally take the form of probability judgments, and why should not this be reflected in the very basis of the pure theory that tries to interpret the realities of economic action?

THE CONSTRUCTION OF STOCHASTICAL FUNCTIONS

Let us define the following two probabilities:

$A(x, y)$ = the probability that A will sign the contract (x, y), if it is offered to him.

$B(x, y)$ = the probability that B will sign the contract (x, y), if it is offered to him.

We can attach these notions to a scheme of economic action which imitates what happens on a bill of exchange. Suppose that the successive transactions will take place on the proposal of a "director," who will repeat the proposal of a given price a certain number of times, beginning with the lowest possible price and continuing the series up to the highest possible one. The dealers are supposed to accept or refuse a unit transaction at every repetition, the functions $A(x, y)$ and $B(x, y)$ representing somehow the force of accepting the proposal for A and B, respectively. A natural hypothesis will then be to put

$$\phi(x, y) = A(x, y) \cdot B(x, y)$$

for the probability that the contract (x, y) will actually be signed by both buyer and seller. The parties are supposed to be acting independently of each other.

As there is a certain frequency distribution (probability) $P(x, y)$ of the size of the quantities x and y which the "director" proposes (very small or very great values being, e.g., quite improbable), the general form of the frequency function of x and y in *signed* contracts will be

$$\psi(x, y) = A(x, y) \cdot B(x, y) \cdot P(x, y).$$

If there is, however—as usually will be the case— a certain (x, y) space where the function $P(x, y)$ is sensibly constant, we may, for the sake of simplicity, neglect this function and study the product $\phi(x, y)$.

THE SIGNIFICANCE OF THE FUNCTIONS $A(x, y)$ AND $B(x, y)$

It seems evident a priori that the probabilities $A(x, y)$ and $B(x, y)$ must depend upon the ophelimities $\alpha(x, y)$ and $\beta(x, y)$, respectively, these functions being supposed to exist. A simple connection would, e.g., be that

$$\begin{aligned} A(x, y) &= 1 \text{ if } \alpha(x, y) > 0; \\ &= 0 \text{ if } \alpha(x, y) \leqq 0; \\ B(x, y) &= 1 \text{ if } \beta(x, y) > 0; \\ &= 0 \text{ if } \beta(x, y) \leqq 0. \end{aligned}$$

In words: a person would certainly sign the contract (x, y) if, as a result of the transaction, his ophelimity is increased, otherwise he would decline.[2]

[2] This corresponds to the case where both parties are "option receivers" to use the terminology of Ragnar Frisch ("Monopole-Polypole—" *Nationaløkonomisk Tidskrift*, Köbenhavn, 1933).

Another point of view would be to put the probability $A(x, y)$ equal to some monotonic function of $\alpha(x, y)$, or quite simply to identify A and α, the distinction being only regarded as a matter of "normalization."

But the most fruitful method of representing the functions A and B will probably be to put them in relation to the subjective estimates which the persons A and B make of the value of the different possible contracts (x, y), when the subjective price is looked upon as *a statistical variable*, characterized by a frequency or probability function $a(p)$. We shall define this function through its connection with $A(x, y)$. We assume that $A(x, y)$ is constant along any straight line $y/x = \text{constant} = p$. Its value here we denote by $A(p)$. A natural suggestion then seems to be to put

$$A(p) = \int_p^\infty a(p)dp,$$

or in words:

The probability that the buyer A will sign the contract (p) is equal to the probability that his subjective price is greater than or equal to p.

For the other person B, who is also supposed to estimate a subjective price with the frequency function $b(p)$, we similarly put

$$B(p) = \int_{-\infty}^p b(p)dp,$$

or in words:

The probability that the seller B will sign the contract (p) is equal to the probability that his subjective price is smaller than or equal to p.

Let ω and π be the subjective prices of A and B respectively. These stochastical variables ω and π will now take the place of the functions α and β, which represent a pre-stochastical phase of the theory. Instead of considering the utility or ophelimity which was taken to determine a person's economic actions, we are studying *directly* his probability of signing at a given price p. The functions $a(p)$ and $b(p)$—or the corresponding functions in the many-dimensional case—replace those functions which in the classical theory are called curves of demand and supply, and their study may give, e.g., the value of y at which a person, for a given x, is most likely to sign a contract of exchange.

THE FUNCTION $\phi(x, y)$

In our case this function takes the form

$$\phi(p) = A(p) \cdot B(p) = \int_p^\infty a(p)dp \cdot \int_{-\infty}^p b(p)dp.$$

If the functions $a(p)$ and $b(p)$ are normal frequency functions, we get simply

$$\phi(p) = \Theta\{h\sqrt{2}(\alpha - p)\} \cdot \Theta\left\{k\sqrt{2}\left(p - \frac{1}{\beta}\right)\right\}$$

where

$$\Theta(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2} dw,$$

and where α and $1/\beta$ are the mean values of the variables ω and π, h and k being the "precision" parameters of the subjective prices of A and B, respectively. The product $\phi(p)$ is a frequency function, characterized, for each set of values, α, β, h, k, by a single point of maximum, which will accordingly define the state *where an exchange is most likely to occur*. This sort of *exchange point* seems more natural than the point of equilibrium defined by the classical theory.

Statistical observations of the behavior of the market ought to make it possible to determine the parameters α, β, h, k, empirically.

NUMERICAL ILLUSTRATIONS

In order to give an illustration, we will fix two prices of 20-years' bills, say 0.6 and 0.25, corresponding to the rates of interest 2.59 per cent and 7.18 per cent. We shall consider two different cases, one where the subjective price of the buyer A is smaller than that of the seller B (a *depression* case), the other when the subjective price of the buyer is greater than the seller's price (a clear *boom* case).

We put accordingly:

Depression Case

$\alpha(x, y) = 0.25x - y$

$i_1 = 7.18\%$; $\alpha = 0.25$; $h = 1.25$; 2.5; 5; 10

$\beta(x, y) = y - 0.6x$

$i_2 = 2.59\%$; $\frac{1}{\beta} = 0.6$; $k = 5$

Vide Figure 1

The functions $A(p)$ for the different values of the precision h are given by dotted curves, but only the branches below 0.5. In the same manner, the ascending function $B(p)$ is given by a broken line. The functions $\phi(p)$ are drawn in full lines.

Boom Case

$\alpha(x, y) = 0.6x - y$

$i_1 = 2.59\%$; $\alpha = 0.6$; $h = 1.25$; 2.5; 5; 10

$\beta(x, y) = y - 0.25x$

$i_2 = 7.18\%$; $\frac{1}{\beta} = 0.25$; $k = 10$

Vide Figure 2

The functions $A(p)$ for different values of h are given as broken lines, the function $B(p)$ for $k = 10$ as a dotted curve; the products $\phi(p)$ as full lines.

As a general result in the case of normal distribution of the variables ω and π we can note that for equal precisions h and k the resulting $\phi(p)$ is a symmetrical function about the mean value

$$\bar{p} = \frac{1}{2}\left(\alpha + \frac{1}{\beta}\right)$$

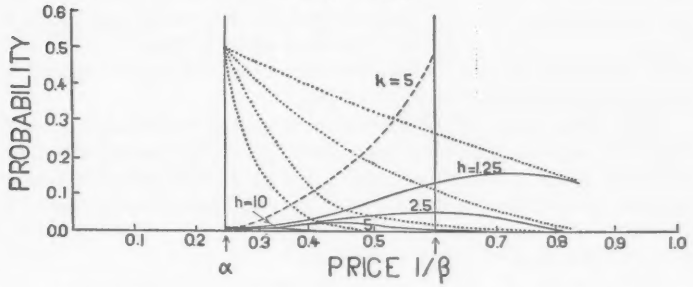

FIGURE 1.—Depression Case. Subjective prices: $\alpha = 0.25$; $1/\beta = 0.6$.

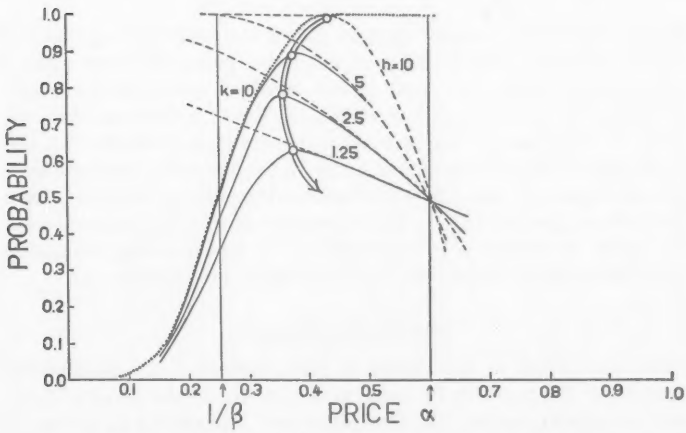

FIGURE 2.—Boom Case. Subjective prices: $\alpha = 0.6$; $1/\beta = 0.25$.

of the subjective mean prices. The line of greatest probability is thus the bisector of the angle between the straight lines $\alpha(x, y) = 0$ and $\beta(x, y) = 0$. This simple result, owing to the precisions (i.e., the convictions) of the buyer and the seller holding themselves in equilibrium, is, as clearly seen from the figures, fundamentally altered when the constants h and k vary against each other.

The depression case, as illustrated in Figure 1, is characterized by small probabilities $\phi(p)$, i.e., a small volume of transactions. For $h=k=5$, the mean price 0.425, at which a contract is most likely to be signed, gives a maximum probability of only 0.012. If h increases or decreases, the point of maximum is *quickly displaced*, and for decreasing h (and k) the maximum probability moves upwards. These features of the depression case seem natural enough if it is considered that the supposed tendencies of A and B are such that the buyer's subjective price (his judgment of the price situation) is low, the seller's subjective price being, on the contrary, very high.

In the boom case, on the other side, when the buyer holds himself to be able to pay a high mean price, $\alpha=0.6$, the seller being inclined to be content with a lower mean price, $1/\beta=0.25$, the probability of a contract being signed will not be far from 1. The curves of Figure 2 give clear evidence of this fact. For equal precision, $h=k$, we have also here a maximum probability at the point

$$\bar{p} = \frac{1}{2}\left(\alpha + \frac{1}{\beta}\right) = 0.425,$$

though perhaps not so well marked. But if one precision, e.g., that of A, varies downwards, the locus of maximum probability will make a complicated motion and come to rest within a rather narrow neighborhood of $\bar{p}$. In the case illustrated in the figure, a decrease of h from 10 to 2.5 will dislocate the point of maximum from $\bar{p}=0.425$ to about 0.375, and it will never pass that point. In the depression case of Figure 1, a decrease of h from 5 to 1.25 displaces the point of maximum from $\bar{p}=0.425$ to about 0.725. In the depression case, the displacement of the point of maximum is accompanied by an increasing maximum probability; in the boom case, by a decreasing probability.

CONCLUDING REMARKS

By introducing in the theory of pure economics the stochastical element, it will probably be made more adapted to the needs of practical economics. Indeed, the interpretation of all reality by means of abstract mathematical schemes must pass through the theory of probabilities, and through the practice of statistical observation. This fundamental fact is reflected by stochastical modifications of theories pertaining to different dominions of thought. As regards economic theory, the stochastical point of view seems to us to be clearly inherent in the very object of the research.

The scheme here proposed must, of course, only be regarded as a modest first venture. Especially, the reader must be warned against

confounding the general theory which eventually could be constructed with the numerical illustrations represented by the accompanying figures. And furthermore it will be necessary to introduce, in the construction of the functions $A(x, y)$ and $B(x, y)$, a series of assumptions concerning the future development, thus classifying the probabilities according to different pictures of the future.

Stockholm

THE EQUILIBRIUM PRICE IN A PERFECT INTERTEMPORAL MARKET

By H. W. Robinson

As soon as the assumption of "perfect foresight" in economic analysis is replaced by other assumptions concerning foresight it becomes very difficult to show how equilibrium can exist and be maintained. For this reason, and for a closer approximation to reality, expectations of individuals and their precise function in the economic nexus are becoming important objects of attention for the economist. It is extremely interesting, therefore, to examine closely the functioning of intertemporal markets, such as the market in cotton futures, in which these expectations are the predominating influence in the determination of prices. It can be argued that, in the forward price, market opinion regarding the future manifests itself in an objective form. But we must investigate the exact relationship which exists between the objective price in the market and the price expectations of individuals before we can use this as a basis for further analysis. The following is an attempt to carry out a simple investigation of this problem.

We must assume, in the first place, a "perfect intertemporal market" for some single commodity, say wheat. This means that, at each point of time, a unique future price exists for the commodity; the future being some constant length of time t beyond each single point of time. The "future" is, therefore, a moving point in time which is always a constant length of time ahead of the present. This unique price is determined by the price expectations of the persons operating on the market and we must bring out clearly the relationship between price expectations (to be defined later) and this unique price. The length of time t is determined by the normal length of time for which futures are quoted, e.g., three months.

We must distinguish between two types of persons operating on the intertemporal market. First, the pure speculators who buy or sell forward according to the divergence between their price expectations and the price quoted on the market. Second, the "hedgers" who produce or purchase wheat and use the market primarily as a means of insurance against risk.

Consider, first, one single speculator. Let this individual possess the following probability judgements concerning the prices of wheat at the future time t (the present being taken as zero time). By "probability judgement" I mean a subjective judgement of the probability of a price lying within a small range δP of the price P as δP approaches zero. This will be based upon all kinds of psychological and economic considerations, such as a natural tendency to optimism or an expectation of

government action, which will vary from individual to individual and which cannot be considered here.

In the following analysis we will use P for price and p for probability throughout. Let the probability of a price P_1 be p_1, the probability of a price P_2 be p_2, and so on. The individual will probably believe that it is impossible for the future price to lie outside some range of prices P_L (the lower limit) and P_U (the upper limit). This means that the probability judgements of P_L and P_U are zero or approximately zero.

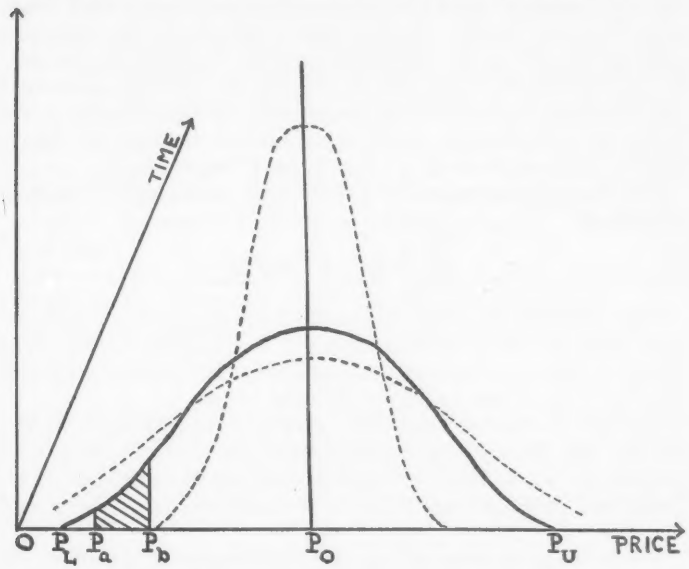

FIGURE 1.—Distribution of expectations regarding future prices.

The whole series of judgements can be represented by a diagram, see Figure 1.

The subjective probability of the price lying between any two prices (e.g., P_a and P_b) is then the shaded area between the ordinates corresponding to these prices.

In making this hypothesis I do not imply that it is possible to measure these probability judgements; I merely suppose them to exist. Neither do I claim that the curves have the particular shape given to them in Figure 1: in fact, since the price can have no upper limit but has a lower limit of zero, it seems reasonable to suppose that the curve will be elongated to the right, i.e., skewed. However, I think that this

fact will be taken into account in the probability judgements of the individual and will only affect the value of P for which p is a maximum. In this case the curve would be drawn symmetrically (as above) because P_0, the maximum probability price, will be regarded as subject to positive and negative errors each of which are equally probable for errors of the same magnitude.

The area of the curve is unity. This is because the speculator considers it certain that the price will be within the range P_L and P_U, the lower and upper limits of the price to him. If necessary a third dimension, T (time), can be added to show variations in the probability judgements with time. In this case the figure is a solid and the plane curve is a cross-section parallel to the price axis and perpendicular to the time axis. If changes in the length of time t for which the person looks ahead (the horizon) are taken into account, then a fourth dimension would be required. Here t the "horizon" is constant.

The mathematical expectation of the price at time t in the future is defined as

$$\sum_{i=P_L}^{i=P_U} P_i \cdot p_i = P_0 \text{ (say)}. \tag{1}$$

This represents the centre of gravity of the plane curve and, in this case, since the curve is symmetrical, it is the price marked P_0 in Figure 1 which has the maximum probability. It is also the price such that the value of gain is equal to the value of loss.

It is obvious that the shape of the curve and the price limits P_L and P_U will vary from person to person (see dotted curves). The more peaked and narrow the curve the more confidence the person has in his judgement, since the range P_L to P_U of possible prices is smaller, and vice versa.

A convenient measure of a person's confidence is, therefore, the standard deviation, σ, of the curve. This is specially suitable in the case of the normal curve of error:

$$p = \int \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(P-P_0)^2/2\sigma^2} \cdot dP.$$

For in this case the ordinate corresponding to P_0 is equal to $1/\sigma\sqrt{2\pi}$, the point of inflexion is distant σ from P_0 and the range is approximately 6σ where σ is the standard deviation. These are well-known properties of the normal curve. We can say, therefore, that the "degree of confidence," which we will call D, varies inversely with σ, the standard deviation: this can be expressed,

$$D = \phi_1(\sigma). \tag{2}$$

We must now consider how this individual speculator will react to any particular forward price quoted on the market at any single moment of time.

If the price is less than P_0, his expectation of the price at time t, then he will buy forward and vice versa. But how far will he speculate? What percentage of the resources at his disposal will be invest on the venture? I should note here that, for simplicity, I am ignoring the possibility of a person's using credit for this purpose, though I think it would make little difference to the argument which follows; percentages would then be capable of exceeding 100.

The extent to which he speculates will depend upon the magnitude of the divergence between the price quoted on the market and his expected price P_0. If this divergence is greater than 3σ he will invest 100 per cent. of his resources[1] becauses prices distant 3σ from P_0 have, in his mind, zero probability; they are beyond the limiting prices P_L and P_U. If, on the other hand, the price quoted is equal to P_0, he will invest none of his resources; the price on the market agrees with his expected price.

Between these limits the percentage of his means invested will vary according to the difference between the price quoted and P_0.

Let the price quoted be P, and let

$$P - P_0 = W. \tag{3}$$

Then we can represent the situation graphically as in Figure 2.

The distance W_0 represents the minimum value of W which must be exceeded before he invests any of his resources at all. This is due to (i) psychological factors which deter him from investing until a certain minimum value is passed, (ii) costs of such transactions as need covering before any profit can be made, and (iii) the value of D, the degree of confidence the person has in his own judgment. We make no assumptions here about the shape of the curve.

We can say, therefore, that if M is the percentage of his means which he invests, then we may assume that M is some function of W; let

$$M = f_1(W). \tag{4}$$

But we must also take account of another factor. When any value of W is given and the form of the function $f(W)$ is known, the percentage of means invested can still vary according to D, the degree of confidence. If D is small then the person has little confidence in his judge-

[1] "Resources" means here the resources available for speculation over and above the normal resources the person requires to live on.

ment and the percentage of resources invested will be small. Similarly if D is large then M is also large. We can draw an analogous curve relating D and M to that relating W and M in Figure 2; see Figure 3.

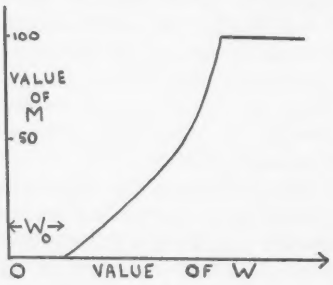

FIGURE 2.—Variation of resources invested with W. (Speculators.)

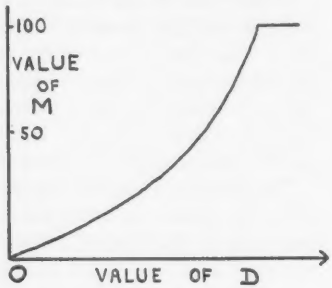

FIGURE 3.—Variation of resources invested with D. (Speculators.)

Here again we can say that M is a function of D. Let this function be,

$$M = f_2(D). \tag{5}$$

From (2) and (5) we have,

$$M = f_2[\phi_1(\sigma)] = g(\sigma), \text{ (say)}. \tag{6}$$

From (4) and (6), on the assumption that M is functionally dependent on W and D, we have that M is a function of both W and σ. We can therefore write

$$M = F(W, \sigma). \tag{7}$$

If we assume that F is linear and that $F(0, 0) = 0$, we have by Euler's theorem,

$$M = \frac{\partial F}{\partial W} W + \frac{\partial F}{\partial \sigma} \sigma \tag{8}$$

$$= \frac{\partial F}{\partial W} (P - P_0) + \frac{\partial F}{\partial \sigma} \sigma \text{ (substituting for } W \text{ from (3))},$$

$$= \frac{\partial F}{\partial W} P - \frac{\partial F}{\partial W} P_0 + \frac{\partial F}{\partial \sigma} \sigma. \tag{9}$$

As M is inversely related to W, since the more "spread out" is the curve in Figure 1 the less is D, then $\partial F/\partial \sigma$ will be negative. Also, as F is linear, $\partial F/\partial W$ and $\partial F/\partial \sigma$ are constants.

Euler's theorem holds true only if the constant term is zero and

this means that if W is zero and σ is zero then M is zero. This is obviously true since, if W is zero the price quoted agrees with the price expected and if σ is zero there is perfect confidence in the expectation, therefore the percentage of resources M invested is zero.

It is interesting to explain what the partial derivatives $\partial F/\partial W$ and $\partial F/\partial \sigma$ mean. $\partial F/\partial W$ is the marginal increase in the percentage of means invested for any given value of W for one individual, that is, the marginal increase for a given price quoted. $\partial F/\partial \sigma$ is the marginal increase in the percentage of means invested for any given value of σ (which measures confidence), that is, the marginal increase for a given degree of confidence. By "marginal increase" is meant the increase due to a small increment in W or σ measured per unit of this increment.

Equation (9) can then be expressed in words: The percentage of resources invested is equal to the marginal increase for the given value of W multiplied by the price quoted P, *minus* the product of the marginal increase for the given value of W and the expected price P_0, *plus* the marginal increase for the given value of σ multiplied by the value of σ (a measure of the degree of confidence). Each product represents, therefore, the share each factor (price quoted, price expected, and degree of confidence) contributes to the total percentage of resources invested.

We can now consider the "hedger." He is in a peculiar position for he has a given quantity (let us say A) of wheat to buy or sell forward. We must assume that this person, like the speculator, has a series of probability judgements which result in some expected price P_0. It is inconceivable that his mind is a perfect blank concerning the future prices of wheat. In the same way as above his judgements can be represented by a curve like those in Figure 1 and he will have a "degree of confidence" which depends upon σ, the standard deviation of the curve. It may be true that, in his case, σ will be much larger because he only uses the market for insurance, but this does not affect the argument.

What happens when any price P is quoted on the market? If W, which is $P-P_0$, is very large the hedger will cease to hedge and take the risk which he would have insured against. In this case he becomes a speculator; he speculates to the extent of the value of the quantity A of wheat. If W is very large there is no reason why the hedger should not buy forward when he would have sold forward and vice versa, and this means that the percentage of A hedged may be negative. If W is very small, he uses the market as a means of insurance and hedges 100 per cent. of A. Between these values of W the percentage of A which is hedged will vary. This can be illustrated diagrammatically. See Figure 4.

Here W_0 is the value of W which must be exceeded before he will take any risks at all. Within this range he prefers to use the market as a means of insurance. This is due to (i) psychological factors which deter him from taking on the function of speculator, (ii) costs of such transactions, (iii) the degree of confidence that he has in his judgement.

Let the percentage of A which is hedged be H. Then H may be assumed to be some function of W. Let this be

$$H = \psi_2(W). \tag{10}$$

We can also argue, as we did in the case of the speculator that, for a given value of W and a given form of $\psi_2(W)$, the value of H depends upon the degree of confidence D. We can give a diagrammatic representation here also. See Figure 5.

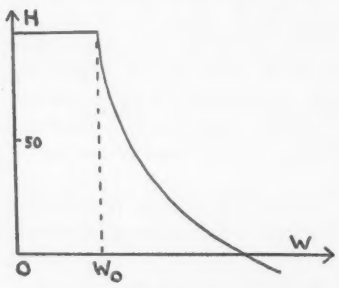

FIGURE 4.—Variation of quantity of wheat hedged with W. (Hedgers.)

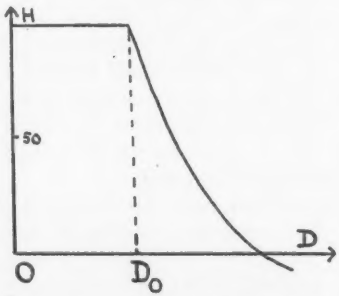

FIGURE 5.—Variation of quantity of wheat hedged with D. (Hedgers.)

The curve remains at 100 per cent. for a considerable value of D (which is shown by the length D_0). For there is a minimum degree of confidence, D_0, which is required before the hedger becomes a speculator.

That is, H is also a function of D. Let $D = \phi_2(\sigma)$ for hedgers and let $H = \theta(D)$ for a given value of W, then,

$$H = \theta[\phi_2(\sigma)] = \psi_1(\sigma) \text{ say}. \tag{11}$$

It follows from (10) and (11) that H is a function of both W and σ; let this function be

$$H = G(W, \sigma). \tag{12}$$

If we assume, as before, that G is linear and that $G(O, O) = O$, we have, by Euler's theorem,

$$H = \frac{\partial G}{\partial W} W + \frac{\partial G}{\partial \sigma} \sigma \tag{13}$$

$$= \frac{\partial G}{\partial W} (P - P_0) + \frac{\partial G}{\partial \sigma} \sigma,$$

$$= \frac{\partial G}{\partial W} P - \frac{\partial G}{\partial W} P_0 + \frac{\partial G}{\partial \sigma} \sigma, \tag{14}$$

where $\partial G/\partial W$ and $\partial F/\partial \sigma$ are both constants since G is assumed linear. Since the greater is σ, the less is the degree of confidence, the constant $\partial G/\partial \sigma$ is, in this case, positive. $\partial G/\partial W$ is, however, negative. The remarks made previously concerning the validity of using Euler's theorem in this analysis and the meaning of the partial derivatives and equation (9) can easily be extended to equation (14) which is of exactly the same form.

We must pass on now to consider what happens when we have large numbers of speculators and hedgers in the intertemporal market. Let there be n_1 persons who are pure speculators possessing amounts $m_1, m_2, m_3, \cdots, m_{n_1}$ respectively, to invest in whole or in part.

Similarly let there be n_2 persons who are hedgers and who have to buy or sell quantities $A_1, A_2, \cdots, A_{n_2}$ of wheat in the future.

Let us also give the functions F and G subscripts to represent the different forms of the functions for different persons. Thus speculator number one has m_1 to invest and his function is F_1. Similarly for any given value of P we can distinguish the different values of W and σ which belong to different people by giving them subscripts also. This amounts to giving P_0 a different value for each person and therefore P_0 can be given subscripts in the same way.

In the following analysis P is assumed to be the equilibrium price and is, therefore, treated as a constant.

Now when W is positive for a person it means that P is greater than P_0, therefore the person sells forward if he is a speculator and hedges completely if he is a hedger who wants to sell in the future or even sells more than the quantity A if W is large enough. If W is negative the speculator buys forward and the hedger *tends* to become a speculator by refraining from hedging his commitments completely. The hedger who wants to buy in the future will act in the opposite way to the one considered above. Thus at any price P quoted in the market some people wish to buy forward and some to sell forward. If, for purposes of analysis, we regard a purchase as a *positive* transaction and a sale as a *negative* transaction, then the total of all transactions at a moment of time must be zero. This is a necessary result of the fact that, at each moment of time, there must be a corresponding buyer

to every seller and therefore the total sales must equal the total purchases. The hedger number j has a quantity of wheat A_j and hedges a fraction, H_j, of it; the quantity hedged is thus H_jA_j. The value of the quantity A_j of wheat at the price P is A_jP and we will call this X_j. It is the value of the wheat which will be possessed or required in the future by hedger number j, and is now in the same units (monetary)[2] as the quantities m_1, m_2, etc. We have

$$X_j = A_jP. \tag{15}$$

Then at any price P the hedger will hedge H_jA_j of wheat, and, by substituting for A_j from (15), this is

$$\frac{H_jX_j}{P}. \tag{16}$$

At this price the speculator number i will invest M_im_i.

The total transactions at any moment are, by summing the transactions of all persons,

$$\sum_{i=1}^{i=n_1} M_im_i + \sum_{j=1}^{j=n_2} \frac{H_jX_j}{P}. \tag{17}$$

When P is the equilibrium price no more transactions are being made and the expression above is zero. It must be noted that H_jX_j and M_im_i are positive or negative according to whether the person buys or sells. Substituting for H_j and M_j from (8) and (13) we obtain

$$\sum_{i=1}^{i=n_1}\left(\frac{\partial F_i}{\partial W_i}\, W_i + \frac{\partial F_i}{\partial \sigma_i}\, \sigma_i\right) m_i + \sum_{j=1}^{j=n_2}\left(\frac{\partial G_j}{\partial W_j}\, W_j + \frac{\partial G_j}{\partial \sigma_j}\, \sigma_j\right)\frac{X_j}{P},$$

i.e.,

$$\sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial W_i}\, W_im_i + \sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial \sigma_i}\, \sigma_im_i + \frac{1}{P}\sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial W_j}\, W_jX_j + \frac{1}{P}\sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial \sigma_j}\, \sigma_jX_j = 0. \tag{18}$$

Hence P, the equilibrium price, is equal to

$$-\frac{\displaystyle\sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial W_j}\, W_jX_j + \sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial \sigma_j}\, \sigma_jX_j}{\displaystyle\sum_{i=1}^{i=n_2} \frac{\partial F_i}{\partial W_i}\, W_im_i + \sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial \sigma_i}\, \sigma_im_i}. \tag{19}$$

[2] Money is regarded only as a "numeraire," i.e., neutral.

We must now examine the signs of the various quantities involved. Consider the speculators. If W is positive, then $P>P_0$ and the speculator *sells* forward. In this case $M_i m_i$, the amount of money he invests, is negative, since selling is regarded as negative buying. That is,

$$\left(\frac{\partial F_i}{\partial W_i}\, W_i + \frac{\partial F_i}{\partial \sigma_i}\, \sigma_i\right) m_i < 0,$$

by substituting from (8) for M_i.

Now $$\frac{\partial F_i}{\partial W_i} > 0;\ W_i > 0;\ \frac{\partial F_i}{\partial \sigma_i} < 0;\ \text{and}\ \sigma_i > 0;$$

hence the sign of the bracket is positive or negative according to whether

$$\left|\frac{\partial F_i}{\partial W_i}\, W_i\right| \text{is} > \text{or} < \left|\frac{\partial F_i}{\partial \sigma_i}\, \sigma_i\right|;$$

i.e.,

$$\left|\frac{W_i}{\sigma_i}\right| > \text{or} < \left|\frac{\partial F_i}{\partial \sigma_i} \div \frac{\partial F_i}{\partial W_i}\right|.$$

As $\partial F_i/\partial W_i$ and $\partial F_i/\partial \sigma_i$ are constants, this criterion becomes

$$\left|\frac{W_i}{\sigma_i}\right| > \text{or} < K_i, \text{ a constant.} \tag{20}$$

Thus, as $M_i m_i$ must be negative, the sign of m_i is negative or positive as

$$\left|\frac{W_i}{\sigma_i}\right| > \text{or} < K_i.$$

Similarly if W_i is <0 then $P<P_0$ and the speculator buys forward and $M_i m_i$ is >0. In this case m_i is positive or negative as

$$\left|\frac{W_i}{\sigma_i}\right| < \text{or} > K_i.$$

Thus the sign of W_i and m_i are not always the same, sometimes the product $W_i m_i$ is positive and sometimes negative.

The same reasoning applies to the hedger. Consider, for a moment, a future buyer. If $W>0$ then $P>P_0$ and he tends to buy less and bear more risk himself. In this case he is, however, *always* a buyer though the extent of his purchases varies. The product $H_j X_j$, which is his investment on the market, is always positive. The sign of X_j depends, therefore, upon a similar criterion to that given above: here

the values of $\partial G_i/\partial W_j$ and $\partial G_i/\partial \sigma_j$ enter instead of $\partial F_i/\partial W_i$ and $\partial F_i/\partial \sigma_i$. We can say immediately that X_j has different signs as

$$\left| \frac{W_i}{\sigma_j} \right| \text{ is } > \text{ or } < \left| \frac{\partial G_j}{\partial \sigma_j} \div \frac{\partial G_j}{\partial W_j} \right|;$$

i.e.,

$$\left| \frac{W_j}{\sigma_j} \right| \text{ is } > \text{ or } < K_j, \text{ another constant.} \tag{21}$$

If the hedger is a future seller he sells and the product H_jX_j is always negative. Hence the sign of X_j depends upon (i) whether the hedger is a buyer or a seller, and (ii) whether

$$\left| \frac{W_j}{\sigma_j} \right| > \text{ or } < K_j.$$

The product W_jX_j will be sometimes positive and sometimes negative.

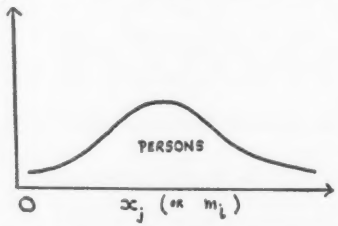

FIGURE 6.—Distribution of resources among persons. Speculators or Hedgers.

Let us now make some special assumptions which can be considered reasonable when n_1 and n_2 are very large numbers:

(I) The magnitudes of m_i and X_j without regard to sign are distributed in a normal (Gaussian) curve about average values. See Figure 6.

(II) Normal distribution of P_{0i} and P_{0j} about an average value of P_0 which is, in equilibrium, equal to P. This amounts to assuming that P, the price quoted in the market, is equal to the average of the individual values of P_{0i} and P_{0j}. P_{0i} and P_{0j} are, of course, the mathematical expectations resulting from the series of judgements shown in Figure 1. This assumes that

$$\sum_{i=1}^{i=n_1} W_i = \sum_{j=1}^{j=n_1} W_j = 0.$$

(III) Normal distribution of $\dfrac{\partial F_i}{\partial W_i}$, $\dfrac{\partial F_i}{\partial \sigma_i}$, $\dfrac{\partial G_j}{\partial W_j}$, $\dfrac{\partial G_j}{\partial \sigma_j}$.

(IV) Normal distributions of σ_i and σ_j.

Assumptions (III) and (IV) amount to assuming normal distributions of the constants in the functions F_i and G_j. This seems fairly reasonable if n_1 and n_2 are very large.

From these assumptions it follows that

$$\sum_{i=1}^{i=n_1} W_i = \sum_{j=1}^{j=n_2} W_j = 0. \tag{22}$$

Also, if the average values of m_i, X_j, σ_i, σ_j, $\partial F_i/\partial W_i$, $\partial F_i/\partial \sigma_i$, $\partial G_j/\partial W_j$, $\partial G_j/\partial \sigma_j$ are

$$\bar{m}_i,\ \overline{X}_j,\ \bar{\sigma}_i,\ \bar{\sigma}_j,\ \overline{\frac{\partial F_i}{\partial W_i}},\ \overline{\frac{\partial F_i}{\partial \sigma_i}},\ \overline{\frac{\partial G_j}{\partial W_j}},\ \text{and}\ \overline{\frac{\partial G}{\partial \sigma_j}},$$

and the deviations from the averages are

$$m_i',\ X_j',\ \sigma_i',\ \sigma_j',\ \frac{\partial F_i'}{\partial W_i},\ \frac{\partial F_i'}{\partial \sigma_i},\ \frac{\partial G_j'}{\partial W_j},\ \frac{\partial G_j'}{\partial \sigma_j},$$

then

$$\begin{aligned}\sum_{i=1}^{i=n_1} m_i' &= \sum_{j=1}^{j=n_2} X_j' = \sum_{i=1}^{i=n_1} \sigma_i' = \sum_{j=1}^{j=n_2} \sigma_j' \\ &= \sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial W_i} = \sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial \sigma_i} = \sum_{j=1}^{j=n_2} \frac{\partial G_j'}{\partial W_j} = \sum_{j=1}^{j=n_2} \frac{\partial G_j'}{\partial \sigma_j} = 0.\end{aligned} \tag{23}$$

Consider the first term of equation (18), namely

$$\sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial W_i} W_i m_i.$$

Now we saw that the product $W_i m_i$ was sometimes positive and sometimes negative. This means that W_i and m_i' are uncorrelated and we can assume $\sum_{i=1}^{i=n_1} W_i m_i' = 0$ Expanding this term by substituting for m_i and $\partial F_i/\partial W_i$ we have

$$\begin{aligned}&\sum_{i=1}^{i=n_1} \left(\overline{\frac{\partial F_i}{\partial W_i}} + \frac{\partial F_i'}{\partial W_i} \right) (\bar{m}_i + m_i') W_i, \\ &= \sum_{i=1}^{i=n_1} \overline{\frac{\partial F_i}{\partial W_i}}\, \bar{m}_i W_i + \sum_{i=1}^{i=n_1} \overline{\frac{\partial F_i}{\partial W_i}}\, m_i' W_i + \sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial W_i}\, \bar{m}_i W_i \\ &\qquad + \sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial W_i}\, m_i' W_i.\end{aligned} \tag{24}$$

But W_i, $\partial F_i/\partial W_i$, and m_i are uncorrelated; from (22) it follows that

$$\sum_{i=1}^{i=n_1} W_i = 0, \text{ and from (23) it follows that } \sum_{i=1}^{i=n_2} \frac{\partial F_i'}{\partial W_i} = \sum_{i=1}^{i=n_1} m_i' = 0;$$

therefore

$$\sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial W_i} m_i' = \sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial W_i} W_i = \sum_{i=1}^{i=n_1} \frac{\partial F_i'}{\partial W_i} W_i m_i' = 0.$$

Hence, as $\overline{\partial F}_i/\partial W_i$ and $\overline{m}_i$ are constants, it follows that each of the sums in (24) is zero, so that

$$\sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial W_i} W_i m_i = 0.$$

Similarly each of the other sums in equation (18) can be shown to be zero. This means that, if the assumptions above were satisfied, and in particular, if the averages of P_{0i} and P_{0j} are equal to P, equation (18) will be satisfied.

From this analysis it seems that, *under the special assumptions made*, the equilibrium price P *can be* the average of the expected prices P_{0i} and P_{0j} of the individuals in the market. This price is no measure of the subjective "probability judgements" of the individuals concerning various prices but it does reduce those subjective judgements to an objective form which has some meaning and which stands in a definite relationship to the mathematical expectations of these individuals, namely the average.

An analysis of the case where non-normal distributions of m_i, W_i, W_j, σ_i, σ_j, $\partial F_i/\partial W_i$, etc., are found would be more difficult and would involve some assumptions (or statistical inquiry) concerning the relative magnitudes and distributions of them. It seems highly probable that if n_1 and n_2 are very large, then even if the distributions of the constants were not normal but were slightly skewed, the equilibrium price would approximate to the average of the mathematical expectations of the n_1+n_2 individuals in the market. How far the distributions could be skewed and how small n_1 and n_2 could be made before the divergence was appreciable is, of course, problematical.

Equation (19) gives an expression for the equilibrium price which is independent of any assumptions regarding normality of distribution of the constants. It is interesting to examine this more closely.

By substituting in equation (18) from equations (9) and (14) we find,

$$\sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial W_i} P m_i - \sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial W_i} P_{0i} m_i + \sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial \sigma_i} \sigma_i m_i$$

(25)
$$+ \frac{1}{P} \sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial W_j} P X_j - \frac{1}{P} \sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial W_j} P_{0j} X_j + \frac{1}{P} \sum_{j=1}^{j=n_2} \frac{\partial G_j}{\partial \sigma_j} \sigma_j X_j = 0.$$

Let

$$S_1 = \sum_{i=1}^{i=n_1} \frac{\partial F_i}{\partial W_i} m_i,$$

$$S_2 = \sum \frac{\partial F_i}{\partial W_i} P_{0i} m_i,$$

$$S_3 = \sum \frac{\partial F_i}{\partial \sigma_i} \sigma_i m_i,$$

$$S_4 = \sum \frac{\partial G_j}{\partial W_j} X_j,$$

$$S_5 = \sum \frac{\partial G_j}{\partial W_j} P_{0j} X_j,$$

$$S_6 = \sum \frac{\partial G_j}{\partial \sigma_j} \sigma_j X_j.$$

The equation then becomes

$$P S_1 - S_2 + S_3 + S_4 - \frac{1}{P} S_5 + \frac{1}{P} S_6 = 0,$$

i.e.,

$$P^2 S_1 + P(S_3 + S_4 - S_2) + (S_6 - S_5) = 0. \tag{26}$$

This is a quadratic in P and, in general, the roots give two equilibrium prices; thus even under our very simple assumptions there is no guarantee that a single price will emerge, for at each moment of time two prices are possible. When $(S_3+S_4-S_2)^2$ is $>4S_1(S_6-S_5)$ the roots of equation (26) are real and there are two equilibrium prices which are possible. When $(S_3+S_4-S_2)^2<4S_1(S_6-S_5)$ the roots are imaginary and an equilibrium price is impossible. We cannot interpret this inequality in terms of expectations and confidence but it is apparent that it means that it is possible to have a situation in which the expectations and standard deviations of the individuals are such that an equilibrium price is prevented from emerging; expectations are

incompatible. When $(S_3+S_4-S_2)^2=4S_1(S_6-S_5)$ the roots are identical and there is only one equilibrium price. This case would seem, however, to require very special conditions and is to be regarded as very improbable. In other words: when we introduce expectations into the determination of market prices it is highly improbable, in general, that a unique price will emerge but multiple positions of equilibrium are to be found and it is difficult to determine which equilibrium position will actually be occupied.

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ON THE THEORY OF CAPITAL: IN REPLY TO MR. KALDOR

By Frank H. Knight

Publication of Mr. Nicholas Kaldor's carefully reasoned article in the July number of Econometrica,[1] criticizing certain papers of mine on capital theory, and the editor's invitation to reply, give me another chance to try to clear up certain controversial points on which there evidently is still disagreement and confusion, even among the most careful students of the problem. As such clarification is the only objective in these pages—not entering into a debate—I am taking somewhat more space than I think would be necessary to meet Mr. Kaldor's criticisms, or his defence of the "Austrian" position, or of any position which might reasonably be called a production-period or investment-period theory.

I

Mr. Kaldor, to my gratification, expresses agreement with so much of what I have been contending for that most of what is now at issue might seem to be chiefly a matter of verbal usage, of what form of expression is best in stating a certain doctrine, or whether one form or another is admissible. In particular, he repudiates in set terms and as definitely as I do Böhm-Bawerk's concept of a period of production, admitting that this is destroyed by the fact of co-operation in further production between capital goods and other agencies, or "produced" and "nonproduced" resources.[2] It is also gratifying not to find in the article any direct reference to an interval between production and consumption (lag of the second behind the first) though the significance of this point will call for qualification toward the end of the present essay. Moreover, Mr. Kaldor admits (in his final paragraph) that the theory of capital in terms of roundaboutness, or "the investment-period concept" in the form and sense in which he still seems to defend it, "could hardly be of any use" "for an analysis of dynamic problems —and specifically of the *par excellence* dynamic problem of the trade cycle." For my part, I have always thought the theory relatively harmless in connection with applied problems. Particularly in the case of Professor Hayek's cycle theory, I have thought—and I know it to

[1] "Annual Survey of Economic Theory: The Recent Controversy on the Theory of Capital," Econometrica, Vol. 5, July, 1937, pp. 201–33. Because these comments are to be published in a later issue of the journal than that in which the article occasioning them appears, it seems advisable in the interest of clarity to quote rather fully from the article in question.

[2] *Op. cit.*, p. 206, at top. See also pp. 224 f. (par. ii) and pp. 225 f. (par. *b*, on durability).

be a common view among his readers of my acquaintance—that in Hayek's exposition one might cross out such expressions as "increased roundaboutness" or "lengthening of the production period," and substitute "increase in the amount of capital" or "further investment" (*ceteris paribus*) without seriously affecting the argument.[3]

Mr. Kaldor further explicitly admits that there is no positive reason for employing the concept of roundaboutness. He says, in italics (p. 231) that "the *Senior-Jevons-Böhm-Bawerkian law of roundaboutness is merely a roundabout way of expressing the law of nonproportional returns*"; and again (not in italics, near the end of the same paragraph) that "the material content of the Austrian theory of capital could be equally well expressed by saying that capital accumulation leads to a reduction in the marginal productivity of the services of those factors whose quantity can be augmented by such accumulation, as by saying that it increases the investment period of the services of those resources whose quantity remains constant." If this is what is meant, I see no reason for saying anything else, and cogent reasons to the contrary.[4]

[3] My own interest has not centered primarily in the applications of capital theory, but in the formulation of a "correct" theory for general purposes. One who is a theorist must assume that such things are of some consequence, without at every step inquiring expressly into the question of what particular falsities of practical conclusion will result from particular fallacies in the statement of general principles. It is in connection with this interest that the present comments and observations are offered to possible readers. Moreover, the fact that an issue is in some sense verbal does not necessarily mean that it is unimportant; it must surely be better for thinkers to say what they mean in the most accurate and intelligible and least ambiguous terms, as well as to mean to say what is in accord with facts or hypothetical assumptions, rather than something else. And in the present case, it will soon be clear that the issues are by no means "merely verbal."

[4] This does not mean that I agree with the alternative statement. The last section of this essay will show that this is far from the truth.

Later on, as we shall see, Mr. Kaldor makes the devastating admission that the view that the diminishing returns from further investment are equivalent to an increase in the period of investment is vitiated as regards any possible application to reality by two considerations. The first is the heterogeneity of the noncapital factor, the second the heterogeneity of the product. The first of these points will be further considered at the end of the present essay, with a view to giving it further development and emphasis. The second objection, I am sure, should be rejected as a factor playing any role in the theory of capital. It is true that there is no absolute, objective measure of output, partial or total, which remains fixed when any change takes place in the proportions of different ingredients in the total. But no economic magnitude is rigorously valid except for an absolutely unchanging system, or for infinitesimal changes in one element at a time at the position of equilibrium for the system as a whole. And economic theory dealing with other and derivative magnitudes has to take as known the quantity of exchange of value of final products, as established by

Let me also state explicitly, before proceeding further, that my own articles on capital theory, cited by Mr. Kaldor, have been devoted especially to the exploration of the conditions under which the quantity of capital "would" bear some definite relation to some quantitatively definable—this does not mean actually measurable—stretch of time which might reasonably be called an "investment period" or "period of production," and to comparing—i.e., contrasting—such conditions with those of reality. I have never denied, I am sure, the possibility of formulating such conditions, and if I have seemed to do so, I hasten to correct the error.

The main conclusion of my argument, developed at perhaps excessive length in former articles (as Mr. Kaldor with some reason complains) is that a theory of the general character of that of Jevons, Böhm-Bawerk, and Wicksell (see note 8) is valid under conditions so far contrary to the facts of real life as to deprive the theory of all interpretive value. Most important are two assumptions: (1) that capital goods are produced by the services of agencies which are not capital goods but belong to a distinct class; and (2) that the individual capital good typically has a history of periodic, rhythmical production, consumption, and reproduction, or investment (of noncapital services) and disinvestment (in consumption). The expressions in parentheses will suggest that the two false assumptions are closely interconnected and that it is the first which is primary, and fatal to the argument.[5]

II

In this section I shall review briefly my conception of the relations between time and the "quantity of capital." Assuming perfect foreknowledge on the part of the investor, and "ideal economic conditions"

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perfect competition among sellers and consumers. (The first "defect in the Austrian capital theory," Mr. Kaldor says, was first pointed out by F. X. Weiss, in an article published in 1921.)

[5] Mr. Kaldor refers (p. 205) to this point as my second argument in connection with the first of three main headings under which my criticisms of the Austrian doctrine can (he avers) be summarized. The first heading itself is that it is impossible to distinguish between permanent and nonpermanent (or "original" and "produced" means of production, or their services (p. 203). In my own mind, it is the impossibility of distinction that is ultimately important; but for the moment we may rest the case on the second fact, that *in any case* the produced agencies are certainly not produced by those which are unproduced, either at a given time or taking economic history as a whole. In most of this essay, down to the final section, it is provisionally granted, or assumed, for the purpose of analysis, that there is a working distinction between capital or produced agencies and another class or other classes. In the final section, this assumption will be examined and reasons shown for its rejection.

in general, the quantity of capital (the value) represented by any instrument or item is equal to, or simply *is*, at the same time, (a) the aggregate net value of all the services invested in it (including its own)[6] and (b) the aggregate net present worth of all the services which it will render in the future. Present worth is found by capitalizing at a rate which is the highest to be had in the field of investment in which the item has been created. The services of an item are of course measured after deducting all "operating expense." This will almost always include some current "maintenance," and may include provision for replacement by an agency of the same form or different form. Replacement is really maintenance of the investment, as I am glad to find Mr. Kaldor agreeing. If perpetual maintenance through replacement is planned for, and if the cost is deducted as an expense in arriving at the current yield, the duration of the future service stream, the service life or durability of the investment, becomes infinite; but the size of the stream is reduced to exactly such an extent that the calculated present worth of the whole stream is unchanged.

In both the invested-value (construction cost) and the present-worth calculations, time enters as a dimension, and interest, compounded continuously, is involved. If enough other things are held equal, in exactly the right way, the amount of capital embodied in, or represented by, any instrument will be greater as *either* the construction period *or* the service life is increased. This is purely because of the role of interest, which expresses the productivity of investment in the system. Of course, neither total computation can vary unless the other varies also, since the two results must always be equal. (Their equality, expressed symbolically in terms of the interest rate as an unknown, for which the equation is to be solved, is the accurate way of stating the empirical (value) facts of interest as a rate of return in a concrete case, i.e., on a particular investment.) I think Mr. Kaldor recognizes this "*ceteris paribus*," and the fact that there are infinitely many other ways of investing more capital, apart from replacing individual instruments with others which cost more because their construction period is longer and/or yield more because their durability is greater.

The real "average investment period" for a *single* instrument or other investment is, as Mr. Kaldor says, a sum of the average construction period and the average service life or durability, both correctly

[6] The investment in an instrument (its cost) *either* includes compound interest on the value of every increment of service of "other agencies" used in constructing it, *or* it includes the accumulated rental value of the incompleted instrument itself at each moment during the construction, integrated over the construction period (or to any arbitrarily chosen instant for which the capital value is being computed).

computed, taking account of interest. It is, however, better described as a properly weighted average of the intervals for which all the separate infinitesimal increments of service value invested in constructing the item remain invested before they are "consumed," with or without provision for replacement, as the item wears out, is used up, or "obsolesces." More or less typically, we may suppose, more costly things take longer to produce, and also last longer. In so far as this is true, the construction period and the service life will be positively correlated, and the quantity of capital will be correlated with each, and with their sum. But this is not universally true, and there is no functional relation, still less proportionality.

All the foregoing reasoning regarding an individual capital good is also valid for any "single investment" however bounded. But two facts must be kept in mind. The first is that in order to compute the value one must have all the data. And in either mode of calculation these include the complete history of the investment, past or future as the case may be, with the dates of all entries, from zero to date and from date to zero, from the first increment of investment to disinvestment of the last increment, either in final consumption or liquidation into another account. One might in theory compute the "investment period" for a national economy or for the world, but only after the close of its history in either case, or after its entire future history became predictable in quantitative detail. The second obvious fact is twofold. First, the "other factors" whose services are invested in the creation of any instrument or investment always include both capital and noncapital agencies indiscriminately. They include all the agencies which in any degree collaborate in the creation of the instrument or investment in question, and in reality this means all the agencies in the "economic system" in which the new investment is carried out; and the only real boundaries of this system under modern conditions are those of the world. Second, in a system which as a whole is not disinvesting, liquidation is liquidation into the system as a whole, not real liquidation, into consumption[7] (see note 12). With these reservations, the investment-period theory may be correct. And it can be modified and so stated as to be valid where capital and noncapital agencies co-operate in the creation of final product. It cannot be formulated, I am convinced, so as to be valid in a system in which capital goods collaborate in any way in the creation of new capital goods or in their own "mainte-

[7] My own slowness in getting this point clear may partly explain the difficulty in following the argument which I have published on the capital problem, of which Mr. Kaldor speaks. For aid in achieving clarity in my own mind, I feel indebted to published articles, and unpublished manuscript, by Mr. Kenneth E. Boulding, and to oral discussion with him (cf. below, note 12, at end).

nance." Such collaboration makes the investment period in each instrument "infinity," or that of the economic system in which it functions.[8]

The essential fallacy in an investment-period theory of capital is thus found in the two "brute facts" already stated, especially the fact that individual capital goods are *never* produced by noncapital resources (even exclusive of the growing individual capital good itself in any case—ignored by Böhm-Bawerk); and indeed in all economic history they never were so produced. In fact, what produces any new capital increment is simply the economic system in which it originates, functioning as a unit. And as a corollary, in any "going" society, every capital good co-operates in its own replacement, by another item of the same kind or of a different kind. In consequence, capital must be treated as perpetually maintained rather than periodically worn out and reproduced—except where one's interest is in the life history of some particular unit, taking the rest of the system as "given," when

[8] See the present writer's "Reply to Dr. Oskar Lange," in *Review of Economic Studies*, June, 1937, and Lange's original article (*ibid.*, June, 1936); also his "Rejoinder" (*ibid.*), in which the main points of my position are conceded.

Böhm-Bawerk's assumptions as to the structure of economic production and the relations of the factors, viewed in the abstract, make one substantial advance over the position of Ricardo (as again I have been rather slow in getting clear). Böhm-Bawerk does not treat wages as fixed by subsistence requirements, or some other noneconomic force, but works out a kind of equilibrating adjustment of both wages and interest on the basis of a conception of competitive process which is partly consonant with reality. Böhm-Bawerk's treatment was criticized and greatly refined by Wicksell, especially in *Ueber Wert, Kapital und Rente* (1893). Böhm-Bawerk rejected the criticisms, and Wicksell (besides other limitations) never carried the theory to the point of recognizing collaboration between capital and labor (noncapital) in the production of capital, as well as in making product. That this would be necessary to make it of any value in interpreting reality, but would make the production-period notion untenable, is the main contention of this *Reply*, as of my earlier essays dealt with by Mr. Kaldor in his article.

Ricardo worked out a theory of distribution for agriculture only, and a theory of the separation of wages and "profit" only for the no-rent margin in agriculture. For industry, he simply carried over the rates of profit and wages as fixed in this connection. In agriculture, he assumed that a given amount of capital (food) "supports" a fixed amount of labor, leaving "profit" as a residual (at the land margin). It is a return per cent per annum on capital because capital is defined as annual income! But of course, too, Ricardo's assumptions vary almost with the passage one may select for examination.

Recently, Professor Pigou has indicated adherence to a production-period view of capital in some form, but I have not succeeded in understanding his reasoning —or in finding anyone else who could do so. See *The Economics of Stationary States*, Macmillan, 1935, and *American Economic Review* for September, 1936. It is noteworthy that Senior, in formulating his third elementary proposition, bracketed labor with "the other instruments which produce wealth," but, as usual with him, failed to follow up his own remarkable flash of insight.

the notion of rhythmical reproduction may have more or less validity.[9] (Even the replacement is not definitely separable from either routine maintenance or operating expense.)

III

Let us now turn to detailed examination of Mr. Kaldor's argument. To begin with, I simply pass over most of the discussion beginning with his Section II and occupying most of Section IV.[10] Regarding Mr. Kaldor's summaries and analysis of my own position, I am glad to note that these are much more recognizable than one comes to expect such statements by critics to be. I may add that while there may be some point to his complaint at the length and intricacy of my argument as published, the *tu quoque* is open as a rejoinder. It usually is!

Mr. Kaldor's clearest indication of the meaning of the period of investment is found in footnotes 19 to 21 on pp. 212–213. In the first of these notes, he expresses his inability to understand my view that the Böhm-Bawerk investment period or "turnover period" is really an accumulation period and not a period of investment. (As noted above, I should have said that it is for him sometimes an accumulation or construction period, sometimes a durability.) For my part I do not understand anyone's having difficulty in understanding that in Mr. Kaldor's own formulations (in the two footnotes next following) his period t is a construction period—or a period of accumulation, if we are thinking of accumulating a money sum for lending or for use in purchasing some security or piece of property, instead of real investment. It is a period of invest*ing*, for individual items instantly consumed, and also for the invest*ment* as a whole, but of course a period of invest*ment*, a turnover period, only for the whole investment, consisting of items in rotation. He distinctly says that his output is a per-

[9] These remarks suggest the fact, perhaps equally important as a source of confusion in interest theory, that the individual saver of "money" usually does not alternately accumulate savings and consume them.

[10] This argument deals with the hypothetical case of especially defined and restricted capital goods (houses) produced exclusively by another factor (labor). As Mr. Kaldor freely admits that such a situation is wholly untypical of real life I frankly do not see why he takes the trouble for himself and his readers to develop it at such length, or what significance he attaches to the conclusion (p. 213) that *under these conditions* "the degree of roundaboutness can be measured by the ratio of the initial or construction cost to the annual maintenance cost." As I shall later insist, it is incorrect to speak of a time period or degree of roundaboutness unless the capital could be economically disinvested and the flow of final product kept up over the interval measured by this quotient. This is rarely if ever approximately the case for a single item, and for society as a whole the notion is fantastic. The quotient is one form of "theoretical" or mathematical turnover period, one among several forms, equally defensible, and to all of which the implied negation of our question as to their "reality" is applicable.

petual flow, which means that the capital never is disinvested, and the "period of investment," in contrast with the investing period (period covered by the process of investing) or turnover period in his special case, is infinite.

In his statement of what "Professor Knight has shown" (p. 212, in text) Mr. Kaldor has misunderstood what I meant to show and misinterpreted my use of the basic equation $a(1+i)^t=b$. His own use of it is correct, but altogether different, and only through correspondence did I, in turn, get his meaning. I set up the picture of an investment made by an individual at the rate of a dollars per year over t years, but who then *stops saving or investing* in any sense, resuming the consumption of his original income of a and also adding to it the net yield of the new investment, treated as a perpetuity. The investment is to be regarded as maintained out of its own yield or as requiring no maintenance, through periodic replacement or otherwise—such a case as digging a well or draining a field. Mr. Kaldor uses the equation to describe the very different case in which a saver uses his a dollars each year for t years in such a way as planting some crop which matures in t years (without tillage expenditure), but who at the end of the t years *continues* to plant a new crop each year while each year gathering and consuming the harvest from the planting of t years earlier. The fact that the formula correctly represents the course of events (the investor's income throughout) in both cases goes far to bring out the issues before us as I see them. Under Kaldor's special assumptions the investment period is (or "would be") quite real. Under mine, there is no period. Neither of these pictures is typical of reality. My position is simply the negative one that reality does *not* correspond with the picture in which the period view would be valid. The facts are overwhelmingly against conceiving of the capital of a system under the form of periodic investment and disinvestment at all, and particularly of noncapital services. At the moment I wish to emphasize that in both pictures t is the time required to build up an investment; under Kaldor's peculiar assumptions, it is *also* the interval of periodic investment and disinvestment of "other" services in a rotating sequence of individual capital goods.[11]

Making the assumptions necessary to validate the conclusion, c/a, capital divided by noncapital input, is indubitably an index of the

[11] The reasoning of Mr. Kaldor's footnote 21 on page 213 puzzled me greatly; but since he has conceded in correspondence that it is confused and erroneous, I need not take space to criticize it, beyond noting this fact.

In my original use of the main equation, I of course meant that a dollars per year are saved *out of other income*. The inclusive rate of saving, including interest on the principal previously saved, up to each moment, will be in excess of a by

capitalistic intensivity of a system. (The main assumption is the distinctiveness of noncapital output; see above, p. 65, note 5.) And just as indubitably, making the *further* assumptions which are necessary, the same ratio measures an "investment period." To repeat, I have never questioned either of these propositions, but only inquired into the nature and admissibility of the assumptions.[12]

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an amount which itself is a function of t, the excess being represented by the formula $a(e^{\delta t}-1)$, where $\delta = \log_e (1+i)$, the instantaneous rate of yield, or "force of interest" corresponding to simple annual interest at rate i, and $a(e^{\delta t}-1)/\delta$ is the amount of capital saved up to any moment t. (It seems to me inadmissible to say, as Mr. Kaldor does in the same note, that the quantity of capital is "the value of current services needed" to construct an equivalent capital good; terms like "current" should be taken strictly, as representing a momentary rate, or intensity, not an amount, especially where time rates cannot be uniform, and an amount is only derived from the rate by integration over time.)

[12] The two main assumptions requisite to the validity of an investment-period theory, as stated a few pages back, might be discussed at some length if space permitted, and stated in various ways. The first, and main one, that capital goods are produced exclusively by noncapital services, obviously implies the distinctiveness of the two categories. For the theory to hold in any simple form, it is necessary to assume further, as Mr. Kaldor rather explicitly does in the footnotes under discussion, (a) that capital goods are created by an "initial" application of other factors, (extending over a negligible interval) and subsequently grow without any further input, and (b) that they are consumed "instantly," (over a negligible interval) when "mature."

For this case, Jevons' formula for the rate of interest $[F'(t) \div F(t)]$, is correct, since differentiating $(1+i)^t$ or $e^{\delta t}$ shows the quotient to be equal to $\log_e (1+i)$ or δ. Or, adapting the formula for interest compounded annually, the numerator would be the annual growth and the denominator the value at the beginning of any year. This is not the situation which Jevons talks about in his text; for the later case—the increasing productivity of a given total quantity of labor as it is spread over a longer period of time—the calculation of interest is considerably more complicated. The relations may be made still more complicated without invalidating the theory, or rendering the computation of interest in terms of an investment period impossible. The details present a mathematical rather than an economic problem.

In the past few years several writers have noted that Böhm-Bawerk's theory is valid only for "circulating capital," meaning evidently consumption goods in process. In this view, "fixed capital," (auxiliary instruments) could be taken into account by treating such agencies as noncapital factors. The essential condition for the computing of a production period is that the investment begin and end at zero value, or be made up of units which do so, without co-operative overlapping. Since in reality this isolation is rare, withdrawals from one capital account being largely payments into another, the average period of investment found will depend within the widest limits on the way in which items are combined or subdivided in setting up the capital accounts. It does not seem admissible to say, without explanation, as does Professor Pigou, (*op. cit.*, p. 53) that "no new principle is involved" in bringing capital into the picture.

After this article is in type and the proof read, I am in receipt of another letter

Mr. Kaldor's Section V (pp. 224 ff.) deals with the relations between the hypothetical case of Section IV and reality and cannot be examined in detail. It seems chiefly remarkable for the recognition of certain facts for which I have contended: (1) That "labor" (noncapital) cooperates with capital both in producing and in using the capital goods, and (2) that it is impracticable or irrelevant to distinguish between (a) provision for replacement, (b) current maintenance, and (c) operating cost, so that all can be lumped together as (the labor or noncapital element in) the annual maintenance of a given service stream. The essential fact is simply that (under the present assumptions) a given amount of labor is combined with a given amount of capital in producing a given steady flow of "product." (This Kaldor takes as "bread" but it should be viewed simply as desired or want-satisfying service, reduced to homogeneity in units of value.)

In Section VI, which he seems to regard as the crucial division of his argument, Mr. Kaldor expounds the theory of capital and interest with reference to a society in which all productive instruments are capital goods (goods augmentable through accumulation, hence not original), but in which these are of two kinds—in his example, "machines" and "slaves." His main reference to the production period in connection with such a system is the statement that there will be two such periods, both of which "will depend on" the ratio of the value of the entire input of the service of the other to the total product of the combination (p. 230). In the previous section, dealing with labor and capital (p. 225), he has told us that an increase in the quantity of machine services will reduce the "quantity of labor input per unit of [product] output . . . which in turn *implies* an extension of the *degree* of roundaboutness" (my italics). This introduces a different "index of roundaboutness" (also mentioned on p. 213)—b/a instead of c/a in the notation of the footnotes discussed above (pp. 69–70), and one which

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from Mr. Kaldor, giving some criticisms of the text above. He points out that in his notation, b is the net yield of the investment, while in mine it is the total income of the investor, including the yield of the investment and the income at annual rate a which he sacrificed during the interval t to build up the investment. This does not affect the substance of the argument, that the position of the investor, before, during, and after the investment, is the same whether the investment is perpetual without maintenance, or with routine maintenance, or only through replacement at whatever interval. Mr. Kaldor also insists that he does not assume t to be a construction period, that is a turnover period. This evidently refers to the later portion of his argument; at this point I was referring only to the assumptions underlying the use of the compound-interest formula, in his footnote as cited. Regarding the turnover period, the later discussion in the present paper will show that the issue is whether the turnover itself, and hence the period, is real.

I like much better *as an index of capitalistic intensivity*. The terms of this ratio would (under the assumption of a real distinction between capital and noncapital agencies and services) be directly measurable and do not involve the definition and measurement of capital.

The new hypothesis of Section VI, however, says nothing about either the construction period or the service life (durability) of individual agencies, and leaves it a mystery as to why capitalistic intensivity should be regarded as corresponding in any way with any investment period, or as to what is meant by the degree of roundaboutness for which the ratio is said to be an index. The ratio would apparently have the same meaning in a system in which both machines and slaves lasted forever and regardless of their origin or what might be known about their past history.

The only investment period to which I am able to attach meaning is one of those already discussed, i.e., either some variant of the Jevons-Böhm-Bawerk-Wicksell average construction period and/or average durability, for all the capital items in a system considered individually, or one of these figures computed for the system as a whole, considered as a single investment. The first Mr. Kaldor has formally rejected at the outset (p. 206, at top) and again in Section V (pp. 225–56). The second concept, I am sure, is not what he means, and I am forced to look for some sentimental or arbitrary reason (reverence for tradition or great men, *à la* Marshall?) for his treatment of his index of capital intensity as an index of roundaboutness or of an investment period.

IV

What is meant by "making use of the concept of an investment period" (Kaldor, p. 213, note) is the final subject for consideration; it raises the main problems of capital theory. On this point also, Mr. Kaldor's position is hard to make out. As already noted, his article ends with an admission that the investment-period concept is of no use in attacking real, "dynamic" problems. So we must look for some function which it may be supposed to perform in theory itself. Near the end, again (Section VI, p. 232), we read: "The purpose of the 'investment period' approach is to reduce the production function to two variables, substituting 'waiting' for the services of all produced (or variable) factors, with interest as the price of 'waiting.' In this way—and only in this way—can *capital as capital* be treated as a factor commensurate with 'labour.' "

Now all this "necessity" seems to me mere confusion and a feature of conventional capital theory, or of its assumed implications, which it is especially important to get rid of. In the first place, capital is not treated "as capital," or as a factor of production, any better by reducing

it to "waiting" when it is not so reducible; and in the second place, it is a serious fallacy to reduce the production function to two variables, or to treat capital as a separate factor of production, or as co-ordinate with labor or with "other factors" defined in any way which is at all consonant with the facts. The issue raised is that more than once referred to above, in conceding *provisionally* that a valid general distinction can be drawn between capital goods and other productive instruments. This I now categorically deny; as already stated more than once, I regard the falsity of this assumption as the ultimate and crucial fallacy in the time-period theory of capital. To deal with the question adequately would involve constructive development of the theory of capital along "sound" lines, which of course is not possible here. But I should like to amplify the sketch given in Section II above, by adding a few general considerations.

The conception of capital in general usage simply is not simple, or unambiguous, or even self-consistent! To begin with, the most puzzling aspect of historical capital-and-interest theory (to me) is that capital has ever been regarded as a "factor of production," "co-ordinate with other factors," however classified, and that "rent" and "interest" have been treated as different "shares," coming from different sources. If any fact of economic life is beyond dispute, the fact that the productivity of capital represents the yield of concrete instruments of some sort surely comes in this category. The yield is rent when it is referred to the agency as a concrete thing; it is interest when referred to the agency as a quantity of capital, or simply to the capital invested or embodied in it. The questions raised are, first, what concrete productive instruments can be said to represent or embody capital, and in what sense; and second, what is the measure of the capital, and its meaning as a quantity.

On the first question, I am really astonished to find Mr. Kaldor (at p. 218) triumphantly accusing me of having "overlooked" a valid distinction among productive instruments, which he says "survives the strictures levelled against the traditional classification," namely, the separation between those which can and those which cannot be produced, or "augmented." So far from overlooking this distinction, I have devoted a very considerable fraction of the argument on whose length and "tediousness" he comments, to showing in detail that such a separation is not valid and cannot be made at all accurately or unambiguously. Indeed, at the outset, Mr. Kaldor has devoted several of his own pages to summarizing and criticizing this argument. It is this argument which, as it appears, especially needs clarification.

The facts seem to me "obvious," properly beyond argument, in the

realm of "look and see." Productive instruments are certainly all "produced" in the general sense that the properties which make them productive instruments have arisen in the economic life of the past; they have taken form or somehow become productive instruments in the course of economic history. Of course *none* are, as to substance, "created" by man. The historical difference among them, in their relation to the capital problem, is in the *degree* to which they have been produced under "economic conditions." But the history itself is irrelevant for distribution theory; the significant difference lies in the way in which they are affected by changes which take place, and the degree in which these changes are controlled by economic considerations; especially, in connection with economic growth, which is the determining consideration in this problem-field, how will the yield of any instrument be affected by the addition to the system of new instruments, of the kinds which will in fact be added? (But of course the future is assumed to involve no definite break with the past.)

The fact which is vital for capital theory—i.e., for clearing up the confusions in accepted doctrine—is that this matter of production under economic conditions, or simply of "economic" production, *is* purely a matter of degree, of which every degree is, as nearly as anyone can tell, equally common in reality; except, perhaps, the pure extremes, which may be viewed as ideal limiting cases. And it is perhaps equally important (in the same connection) that differences in degree show extremely little relation to the traditional tripartite division into three "factors," which, correctly named, would be natural agents, capital goods, and free laborers. It is also noteworthy that the divergence from economic conditions in the origin and growth of productive capacity in various cases or instruments is of various sorts, more or less classifiable, but variously and inseparably mixed in reality. Some productive instruments (and most in some degree) become such by pure "accident," without the intervention of human action at all, or of any action "economically motivated" (see above, and note 13, below). This category merges by degrees into the second; the production of capital goods is an activity affected by any amount of error in planning and execution, from zero to one hundred per cent (in one direction, and many times as much in the other). A third consideration is that economic motives are mixed in all possible proportions with others; these are of many kinds, but may be lumped together here as "sentimental." Discussion of the degrees and kinds of sentimental considerations and of uncertainty and error would obviously call for a volume. For the purposes of such a discussion as this, economic conditions may be defined in pecuniary terms. As regards what we call "production," this

means the use of given resources in such a way as to yield a "maximum" pecuniary return, or return in results measurable in money.[13]

As regards conditions of production and their relations to the capital concept, we may best consider "cases," beginning with the only really troublesome case, that of "labor" (laborers or labor power). *If* we take "objective" market value as a test, as usage certainly suggests, then of course free human beings are not capital goods and not capital. They have no such value, so it can have no relation to cost of production. But this is clearly arbitrary in large measure. On one hand, any type of "property" may be tied up by law, or by sentiment, so as not to be subject to purchase and sale, and in fact a highly determinate and acceptable market valuation is rather the exception than the rule

[13] The other half of economic behavior in an enterprise economy, the use of pecuniary return or income so as to yield "maximum satisfaction," includes both consumption and saving.

Much of the difficulty of economic analysis is due to the fact that we have to define its concepts "at the limit," for "perfectly" economic behavior, at "equilibrium," while in reality most of them become meaningless before the limit is reached. Activity never derives all its significance from any result, still less any pecuniary result, and all activity thought of as economic has results which are in various degrees (not) measurable in money. And all this is essential to our life interests as well as to the unalterable factual conditions of life. On one hand (I am sure) a "Crusoe" (which is another limiting concept meaningless at the limit) would feel a real distinction between economic and other activities, and on the other, a perfectly definite distinction is unimaginable under any conditions. Perfect economy would involve both perfect foresight and perfect efficiency, which is to say omniscience and omnipotence, and both notions, strictly interpreted, are nonsensical. Moreover, action on a basis of mutual foresight of one another's acts is absolutely impossible without collusion. Yet we have to work with the concept of perfect economy.

Another distinction which is necessary but undefinable is that between capital goods, somehow defined, and consumption goods. In strict theory, any "thing" (tangible or intangible) which exists at a moment and which has value is a capital good; the only product actually consumed is a service, which never has "existence." Production is either the rendering of services (which includes maintenance of the agencies rendering them, including replacement in so far as actually involved) or it is adding to total productive capacity. (Whichever it is, the result "obviously" occurs "instantly," as the activity is performed.) Confusion arises from the fact that we (theorists) do not usually apply the economic concepts of capital (i.e., rational) accounting and management in connection with the creation, maintenance, and liquidation of productive capacity in the form of (free) human beings (laborers). This is because usage in the world at large does not do so. But this does *not* mean that rational management is not applied to an important extent, *still less* that results are vitally different from what they would be if it were much more generally and deliberately applied. (We shall return to this topic presently.) It is noteworthy that the older economists, while they did not propose to treat labor power as capital, did assume its value to be strictly determined by cost of production. Cf. also Marshall (*Principles*, p. 660). " . . . wages . . . have much in common with interest on capital."

in connection with the whole class of capital goods. On the other hand, human capacities are "more or less" subject to capitalization, in various ways. The difference is clearly one of degree, not of categorical distinction.

Moreover, as to production under economic conditions; on one hand, most investing is more or less influenced both by sentimental considerations and by ignorance of real economic considerations. On the other hand, it is even more evident that a very considerable fraction of the real investment, or creation of economic productive capacity, taking place in any interval in any modern community, takes the form of "labor power." This applies especially to the factor of "training," but also in no small degree to the number of "laborers." And no one can deny that the distribution of investment between this form and other forms is largely affected, in various ways, by the consideration of relative yield. The choice between investing in personal earning power, say in the form of an engineering education (for oneself or one's heir), and investing in intangible or tangible capital goods in the more usual meaning may be as much a matter of economic calculation as the choice between two forms under the latter heading—or the latter may be as much affected by sentiment and ignorance or prejudice as the former. Again, the difference is clearly one of degree (see note 13).

I will not here go at any length into the case of "land" (natural agents). At this date in history it should be unnecessary to explain that exploration and development, discovery, seizure or conquest—the activities by which "natural agents" are brought within the circle of economic entities—are typically carried on at an economic cost and in large measure under the domination of economic motives. It is, again, "obvious" that, in so far as such activities are carried on under economic conditions, the return from a given expenditure will be the same in this form of investment as in any other carried out in the same market area at the same time. This is to say that the value of the result will be equal to its costs. On an average of what are called "natural resources," there *may* be some difference in the role played by imperfect foresight, or by sentiment. There *may* also be some difference in the "fluidity" of the investment (see below). Or, in each case, there may not. Certainly there is no basis for a classificatory separation; "natural agents" are "capital goods."

It is a familiar fact, as well as clear from the foregoing, that capital as a quantity may be looked at from three standpoints: what has been put into any item, what is in it at any moment, and what can be got out. Under conditions of perfect foresight, the quantity according to the first and second modes of measurement would be identical; the amount invested in any agency would be equal to the discounted value

of its future service stream. (This fact is the basis of a definite calculation of the rate of interest.) Of course, this is not accurately true in reality, where conditions are changing and foresight proverbially imperfect. The difference is closely connected with the third basis of quantification, the "fluidity," of investments once made, when conditions change. And this "fluidity" is another important factor in the ambiguity of the general conception of capital and of the degree in which any concrete thing is capital, as well as in the amount of capital it represent or contains. Under "ideal conditions," again, every investment would be unaffected by this consideration; it would have whatever fluidity it needed for perfect adaptation to any change which actually occurred.

But we cannot say that the quantity disinvestable would be the same as that originally invested. The concept of fluidity is ambiguous. What can be taken out of an investment is typically different from the standpoint of the individual owner and that of the economic system. The individual owner may (and ideally always would, though he would have no interest in it) confront perfect fluidity, through sale to some other individual; but from the standpoint of the system there may well be, even under ideal conditions, almost complete fixity, no possibility of real disinvestment. Real fluidity, from the standpoint of the system, is infinitely various, depending on the nature of the change, real or hypothetical, in conditions, in connection with which disinvestment is to be considered, and on the nature of the investment.

Mr. Kaldor is (I say) clearly and egregiously wrong in holding (see his pages 218 ff.) that diminishing returns from capital implies changes in proportions between capital as a "factor of production" and (an)other co-ordinate "factor(s)." It is the cornerstone of his argument, and a cardinal error of the whole time-period conception. In most cases, neither the cost nor the possibility of exact reduplication is in question in determining capital yield or quantity. The reason is simply that reduplication is not what would happen, not the form that capital growth would take, under most circumstances in real life, given perfect freedom of choice—even apart from new inventions or changes in wants. In an extreme case, such as a hydroelectric plant or a railway system, the very notion of physical (Mr. Kaldor says "identical"—p. 219) reduplication is absurd. The presumption is that under stationary conditions new additions to the capital of a system would represent every degree and innumerable kinds of physical similarity and dissimilarity to those previously existing, and would be infinitely various and complex in their relations of complementarity and competition to the latter. As to the amount of capital in any investment (recoverable for consumption or for some other productive use by the individual

owner) under any conditions it is the (minimum) cost of producing *any* new addition to the total capital of the system which will yield the same net perpetual income that is expected from the investment in question (with or without changes in its form). Neither the physical character of the new "instrument" *nor that of the product it will yield* is in question at all.[14] When an addition is made to the total capital of a system—by sacrificing consumable product at some rate, in value units, over some interval—increase in the stock of any class (or classes) of productive agencies previously existing will normally account for a very small fraction of the change. On the contrary it would be approximately correct to make the antithetical assumption that there will be no exact reduplication, or "augmentation," of anything. The addition, produced by the system as a whole, is added to and used as a part of the system as a whole. On one hand, the capital "factor" in a productive system as it stands at any time has only the meaning already explained. What has actually been invested does not affect it; cost of reduplicating an agency or its service can at most set an upper limit to value; and what can be disinvested comes into the reckoning only indirectly, in connection with changes expected or allowed for. The total capital in a system means simply the aggregate present worth of all its capitalizable income items, however defined, taken one at a time and each in turn treated as "marginal," with all the rest of the system treated as "given." The capitalization rate is that which at the moment measures the yield of new investment at the margin of growth. The choice of items to be capitalized is certainly not affected by their origin or past history.

For general purposes, it is no doubt expedient to exclude free laborers, but the reasons for the exclusion must be kept in mind and their logical limitations, which are quite sweeping, must be clearly recognized. Any addition made to a production system under the condition that wants (satisfaction functions) and technology (production func-

[14] This cost of construction of the pecuniary equivalent of an agency is obviously mathematically identical with the capitalized value of the expected future services of the agency, or of the investment in it transformed as occasion may call for and as its own character makes possible. Under "ideal conditions," to repeat, this magnitude is identical with the amount previously invested in the item. But *even under perfect foresight* that can be accurately true only for an investment of negligible magnitude, added to a system in which no other change is taking place.

Most forms of investment as they stand at any time are relatively fluid, or disinvestable, "at the margin," i.e., if disinvestment does not proceed too far. What would happen in connection with progressive net disinvestment in a system as a whole would depend on so many conditions not predictable in advance, especially as to "rationality" and the spirit of enterprise, that I do not consider speculation on the problem to be worth while.

tions) are stationary (in a sense compatible with growth in productive capacity) is an addition to capital in so far as it is added under economic conditions. The matter of the possibility of "capitalizing," through exchange in the market, after it is finished, is not in point, under this stationary assumption. As additions are made, i.e., as growth goes forward, both the physical character of the existing system and that of the additions will continuously change. There is no unambiguous measure either of the addition made or of the aggregate to which it is added, except "value," which is a matter of product prices and the interest rate. The change is qualitative except as measured in satisfaction units. There is no measurable change in the physical proportions of anything to anything else. Labor is no real exception, for (a) as already noted, any increase in labor capacity is also the result of investment; and in addition, (b) the character of the capacity added will be heterogeneous with what existed before, and in its own composition, and both will change continuously as growth goes forward. It can never be admissible to say that labor (capacity or service) in any physical sense has grown in any numerical ratio, or in any numerical proportion to anything else, though real changes may diverge from such a postulate in different degrees.

It is true that nonreduplicability in existing agencies is a factor in the diminishing returns from investment; but it is a relatively small factor, and operates in different cases in widely different degrees. The main fact lies much deeper, in the nature of products and their "utility," in relation to economic growth; it raises fundamental difficulties in the notion of a satisfaction function. Perhaps some awareness of these difficulties underlies Mr. Kaldor's reference to the nonhomogeneity of products. The fundamental fact in question is that what has been said of productive instruments is true in large part because it is true beforehand of products themselves. When the income of an individual increases, in units of fluid purchasing power which he is free to spend as he pleases in a given price situation, he will normally wish only within narrow limits to increase his consumption of products previously purchased. Much more he will wish to add new products to his consumption budget: but, again, he will not stop with this, but will to a considerable extent reduce the expenditure on products previously used, or even drop them entirely, and substitute new items. Even if increased production took the form of increasing the output of the same identical goods and services, without change in proportions, and if these were produced by use of the same productive agencies in the same proportions, all agencies being freely augmentable, investment would still be subject to diminishing returns, because of the diminishing utility of total economic income to the individual.[15]

[15] A somewhat fuller treatment of these points was published by the writer in

In brief (and in conclusion): The entire notion of "factor of production" is an incubus on economic analysis, and should be eliminated from economic discussion as summarily as possible.[16] Its place is in the history of dogma, specifically of the classical distribution theory, which, with all respect to the authors and their many sound insights, was as a whole a tissue of fallacy and irrelevance. It did not approach the problem of distribution as a phenomenon of the competitive (or monopolistic) evaluation of services—the only approach which makes sense—but as one of "cutting the pie" among social classes. (See note 8, third paragraph.) And in consonance with truth, no different or less "strong" language can be used of the notion of capital as representing, or as measured by, any "waiting period" between production and consumption, to which we must turn to find the only real meaning of the investment-period theory.

Under the hypothetical conditions necessary to make the investment period itself real it would also measure a real interval of lag of the consumption of the fruit of any increment of productive activity behind the expenditure of the service. (This lag theory Mr. Kaldor himself apparently rejects.) If capitalistic production really had approximately or in principle the general form of some of the theoretical constructions used to illustrate the investment-period theory, the theory would be valid. If, for example (in accordance with Kaldor's assumptions on pages 212 ff. and notes), a Crusoe lived by planting a turnip each day and harvesting and eating one each day, with turnips growing at a definite rate and each one being allowed to grow for a certain number of days, these notions would be very meaningful. Under these conditions

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the article (review article) already mentioned, entitled "Issues in the Economics of Stationary States," in the *American Economic Review*, September, 1936. The paragraphs in question, it may be noted, were abridged from a longer manuscript, so far withheld from publication because I do not feel that I have reached a final position on the questions raised. It was my hope that what was published might stimulate some discussion which would shed light on the issues. The same hope may be recorded in connection with the observations offered above.

[16] The notion of capital, "as capital," is historically a phenomenon of borrowing and lending money. In modern economic analysis, and specifically in distribution theory, it is, in contrast, essentially bound up with what is called (miscalled) "dynamics"; i.e., it is associated with growth in productive capacity, but also with transfer of productive capacity from one use to another (in response to some change) where a change in form is involved and not merely the movement of given concrete instruments. It is in this last connection that the importance of "fluidity" is especially great. In an unchanging system there would be no occasion for thinking of "capital," since the purposes of a loan would be more simply effected by a lease, and there would be no new investment, and no transfers, either from one use to another or from one owner to another. Ordinarily, only productive capacity which is prospectively in process of investment, not yet committed, is perfectly fluid, and hence is "capital" in the strict, ideal sense.

any increase or decrease in the rate of noncapital input would be reflected in a corresponding change in output at the end of the given number of days. Moreover, if economic life in any society had the form of carrying on any number of lines of production and consumption of this technical character, the time required and other conditions of a transfer of production from one line to another would be calculable from the production-period investment periods in the two lines; but an average for all lines would seem to have no use.

No determinate interval of lag between production and consumption actually has any reality whatever. We cannot talk sense about the time that would be required to build up the existing economic world; and if noncapital input, in any possible definition, were halted, a large part of the population would be starving in a few days—and no small fraction fighting for water in a few hours at most! A given change in the input of noncapital services, defined again in any possible way, may be reflected in a change in output after any interval whatever. All that can be said is that if any increment of investment is economically carried out, the amount and distribution in time of the resulting increase in total yield will bear to the amount and time-distribution of the investment some relation in conformity with the going rate of interest and the compound-interest law (and reciprocally for any disinvestment).[17]

The notion of an investment period must simply be dropped, as having no reality or meaning under conditions resembling those of real life, or it should be used only in connection with a careful statement of the conditions under which it would be valid, contrasted with those of reality. The notion of capital intensivity can hardly be dismissed as meaningless, but no "index" can have any high degree of accuracy. The ratio of total interest, or payment to "property," to total wages, in the system, would be its measure, but neither of these magnitudes can be known or even defined at all definitely. I should not even say that it is never possible to recognize differences in the "proportions" of "natural resources" to labor capacity, or to "artificial" capital; but the assignment of any numerical value to either ratio would seem to be rash in the extreme, and to be avoided in work pretending to a scientific character.

The University of Chicago

EDITOR'S NOTE: A *Rejoinder* is in preparation by Mr. Kaldor but was not ready in time to be included in this issue.

[17] It is disappointing to find a work on economics so sound in theory and clear in exposition as Mr. E. H. Phelps Brown's *Framework of the Pricing System* marred by apparently uncritical acceptance of the notion of capital as essentially related to an "interval between exertion and fruition." *Op. cit.*, pp. 178, 179.

NOTE ON PROFESSOR FRISCH'S "THE PROBLEM OF INDEX NUMBERS"[1]

By A. L. Bowley

The relationship between Frisch's formula (6.6) (p. 28) and that given by me in the *Economic Journal*, 1928, p. 226 can be exhibited more clearly as follows:

Frisch's notation is: p_0, p_1 are prices of a commodity at two dates; q_0, q_1 are quantities bought; $\bar{q}_1$'s are the quantities necessary to give the same satisfaction at prices p_1 as at prices p_0; ω_0, ω_1, and $\bar{\omega}_1$ are the "nominal marginal utilities" of the purchases q_0, q_1, $\bar{q}_1$.

I identified $\bar{\omega}_1$ with ω_1. The difference between the formulae is entirely due to a value of γ, where $\bar{\omega}_1=\omega_1(1+\gamma)$. Frisch's λ equals $1+\gamma$.

Write

$$\omega_0 = \mu\omega_1.$$

Write

$$B = \frac{\sum (q_0 + q_1)p_1}{\sum (q_0 + q_1)p_0},$$

which Frisch terms the "Bowley" formula.

Actually the complete formula I gave for the "indifference index" was:

$$\begin{aligned} I &= \{\sum (q_0 + q_1)p_1 + \mu\sum (q_0 - q_1)p_0\}/2\sum q_0p_0 \\ &= B + (I - \mu)\sum (q_1 - q_0)p_0/\sum (q_1 + q_0)p_0, \end{aligned}$$

and I argued that the second term would be small under certain conditions, which were found to hold in the budgets examined.

Frisch gives:

$$\begin{aligned} P_{01}(I_0) &= \frac{\sum (\omega_1q_0 + \bar{\omega}_1q_1)p_1 + \omega_0\sum (q_0 - q_1)p_0}{(\omega_1 + \bar{\omega}_1)\sum q_0p_0} \\ &= \frac{\sum (q_0 + q_1)p_1 + \mu\sum (q_0 - q_1)p_0 + \gamma\sum p_1q_1}{(2 + \gamma)\sum q_0p_0} \\ &= I + \gamma\frac{1}{4\sum q_0p_0}\sum (q_1 - q_0)(p_1 - \mu p_0), \end{aligned}$$

when γ^2 is neglected.

The term involving "bias" is small if γ is small and the correlation between change of quantities and the difference between the price ratios and μ is not large. This correlation is normally negative, and γ is positive (as Frisch states); so that the bias in my complete formula is normally to overestimate $P_{01}(I_0)$.

[1] Econometrica, Vol. 4, Jan., 1936, pp. 1–38.

I do not feel certain, however, that the γ term is not of an order already neglected with the third-order terms of the Taylorian expansion.*

London School of Economics

* In my note, "The Double-Expenditure Method," in this issue of ECONOMETRICA are given numerical examples illustrating the degree of approximation obtained on the one hand by using Edgeworth's formula (to which Bowley's formula leads when μ is not known and hence B the only term retained in the above complete formula) and on the other hand by using the double-expenditure method (which follows from a somewhat different and, I believe, more correct expansion using, however, Bowley's main idea). It should be noted that in these examples the Edgeworth-formula results are obtained by inserting in the formula the theoretically exact commodity quantities in equivalent points (which of course in practice are not known), while the double-expenditure results are obtained by using only observable Engel curves. In reality the comparison between the results is therefore even more favorable for the double-expenditure method than the mere magnitude of the results indicate.—R. F.

THE DOUBLE-EXPENDITURE METHOD

By Ragnar Frisch

In my survey on index numbers in the January, 1936 issue of Econometrica I developed—following a suggestion by Professor Bowley—*the double-expenditure method* which aims at giving an approximation to the exact indifference-defined price index. The data needed for the application of the method are a set of Engel curves for the base situation and a set of Engel curves for the object situation. An Engel curve in a given situation is a curve showing how the consumption (or sale, or production, etc.) of a given commodity changes as a function of the total expenditure. In the present note the closeness of the approximation will be considered.

In order that it shall be possible to use the double-expenditure method, the Engel curves in the two situations must pertain to *the same kinds of goods*, in other words the same type of data is needed as in the case of the usual index numbers, such as Laspeyres', Paasche's, etc. The problem of structurally different markets is thus not covered either by the usual formulae or by the double-expenditure method. But if the kinds of goods are actually the same in the two situations, the double-expenditure method seems to give an excellent approximation, if one can judge from the constructed examples given below.

In short the method can be described thus. Let $q^1, q^2, \cdots, q^n$ be the quantities and $p^1, p^2, \cdots, p^n$ the prices, the different situations being denoted by subscripts. To a given price system corresponds an *expansion* path through the q-space, that is, a path along which a typical individual moves as his total expenditure increases. The change that takes place in each q-co-ordinate during this movement is given by the Engel curve for the good in question. Consider the two expansion paths corresponding to the situations 0 and 1. In each point on the 0-path we may ask two questions: How much will it cost to buy this quantity combination at the prevailing 0-prices? And how much will it cost at the prices of the other situation, the 1-prices? It will cost $\sum p_0q_0$ and $\sum p_1q_0$ respectively. The product of these two quantities, i.e., $D=(\sum p_0q_0)\cdot(\sum p_1q_0)$ is the *double expenditure* along the 0-path. It may be computed for each point on the 0-path. Similarly the double expenditure along the 1-path will be $D=(\sum p_1q_1)\cdot(\sum p_0q_1)$. Now the double-expenditure method consists in assuming that those points on the two paths are *equivalent* (i.e., have the same indicator of total utility) which have the same value of the double expenditure S. In other words the double expenditure is taken as an indicator. The ratio $\sum p_1q_1/\sum p_0q_0$ taken between any two points that are indicated as equivalent by this criterion will be an approximation to the exact

indifference-defined index. The reason why this ought to give an approximation is explained in the ECONOMETRICA survey referred to.

To give an idea of the closeness of the approximation obtained in this way I have had some numerical examples computed. I shall not discuss the reason which led to the adoption of the special form of the indicators (total-utility functions) used in the examples. This must be reserved for another occasion.

First Example: Two commodities were considered, their quantities being denoted by q^1 and q^2. The indicator was taken as

$$I(q^1q^2) = q^1 + \log_e q^1 + q^2 + \log_e q^2. \tag{1}$$

The partial derivatives (the marginal utilities) are

$$I^1 = 1 + 1/q^1, \qquad I^2 = 1 + 1/q^2.$$

Hence the equation of the expansion path in the situation t is

$$\frac{1 + 1/q^1}{p_t{}^1} = \frac{1 + 1/q^1}{p_t{}^2}.$$

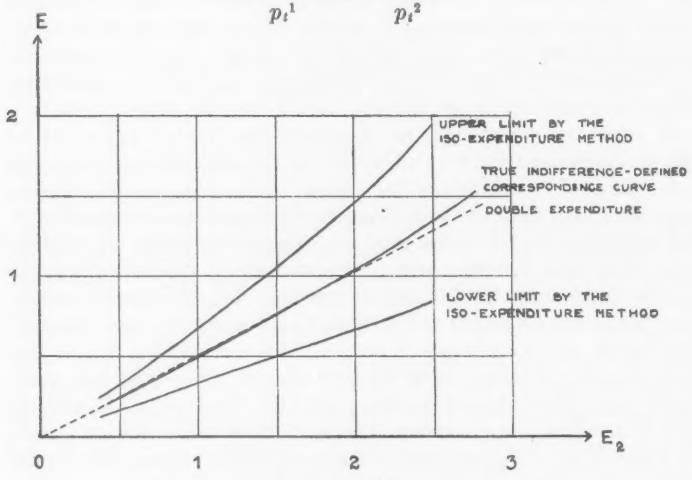

FIGURE 1.

The total actual expenditure in a point (q^1, q^2) on this path is

$$E_t = p_t{}^1q^1 + p_t{}^2q^2.$$

If the prices $p_t{}^1$ and $p_t{}^1$ are given, the double expenditure along the t-path can be computed as a function of some conventionally chosen

parameter, for instance, the actual expenditure E_t. This was done for combinations of the following three paths:

0-path	1-path	2-path
$p_1=1$	$p_1=1$	$p_1=1$
$p_2=1$	$p_2=1.2$	$p_2=5$

In Figure 1 are shown the true correspondence curve between E_0 and E_2 (defined by exact equivalence) and the correspondence obtained by the double-expenditure method. It will be seen that even in this case where the price situation is exceedingly skew—one price having quintupled while the other is unchanged—the approximation is very close.

For comparison and as an expression for the highly skew price change are also computed the upper and lower limits by Dr. Staehle's iso-expenditure method. (Staehle's method is based on a combination of the Haberler and Konüs limits. See the ECONOMETRICA survey.)

TABLE 1

	$I=-2.818576$	$I=-0.38629$	$I=0.88655$
Exact indifference-defined index	2.05	1.94	1.84
Double expenditure	1.90	1.90	1.90
Laspeyres	3.00	3.00	3.00
Paasche	1.58	1.48	1.43
Fisher's Ideal	2.18	2.11	2.07
Edgeworth	2.45	2.40	2.38

We may further compare with the results obtained by some of the usual formulae, say Laspeyres', Paasche's, Edgeworth's, and Fisher's Ideal formulae. To apply these we must decide on what *quantities* to work with. The quantities change along the paths. To let the formulae work under the best possible conditions, let us choose three indifference levels and insert in the formula the *exact* quantities which in our constructed example are known. The result is given in Table 1. From the great discrepancy between Paasche's and Laspeyres' formulae (even having inserted the correct quantities) it is seen that we have a situation that ought to be extremely difficult to approximate. Of the usual formulae Fisher's Ideal gives by far the best result, but the double-expenditure method gives in most cases an even better approximation. It is indeed so close that it is sufficient for most practical purposes. And then it must be remembered that the result in the case of the double-expenditure method *has been obtained without assuming the knowledge of the exactly equivalent quantity* combinations that were used in Fisher's and the other formulae.

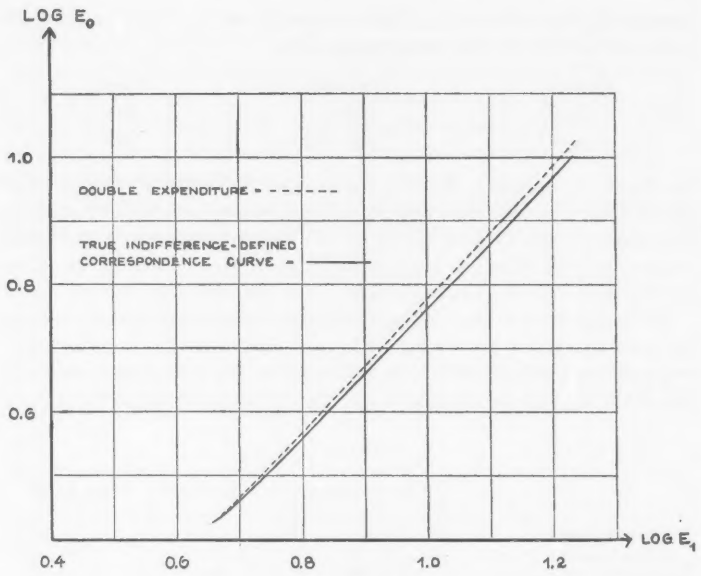

FIGURE 2.

Comparing the situations 0 and 1 we get a curve that is not distinguishable from the true curve if drawn on a reasonable scale. In this case the two limits given by the iso-expenditure method are also sufficiently close for practical purposes.

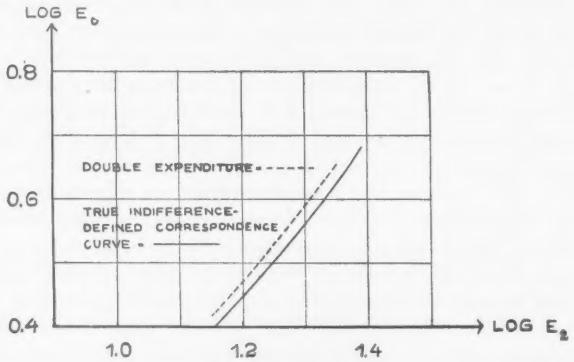

FIGURE 3.

Second Example: In this example the indicator was taken as

$$(2) \qquad I(q^1q^2) = q^1e^{2/q^1} - Ei\left(\frac{2}{q_1}\right) + 2q^2e^{3/q^2} - 6Ei\left(\frac{3}{q_2}\right),$$

where the Ei-function is defined by

$$(3) \qquad Ei(x) = \int_{-\infty}^{x} \frac{e^{\xi}}{\xi}\, d\xi.$$

Here the price situations considered were

0-path	1-path	2-path
$p_1 = 1$	$p_1 = 1$	$p_1 = 1$
$p_2 = 2$	$p_2 = 4$	$p_2 = 16.03.$

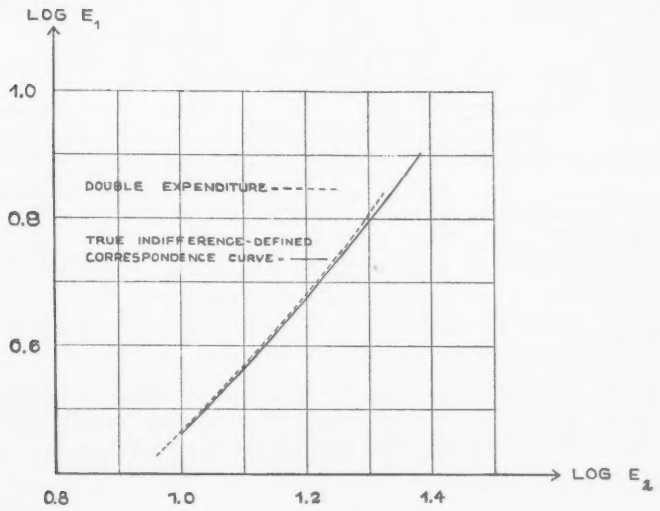

FIGURE 4.

The correspondence between E_0 and E_1, E_0 and E_2, E_1 and E_2 respectively, as determined by the double-expenditure method and by exact indifference will be found in Figures 2, 3, and 4. In all these cases the closeness of the approximation is sufficient. A comparison of the price indices as determined by the various methods (assuming again in the case of Laspeyres', Paasche's, Edgeworth's, and Fisher's formulae the exact quantities) is given in Table 2.

TABLE 2

0-PATH AND 1-PATH

	$I=-24$	$I=2$	$I=10.1$
Exact indifference-defined index	1.72	1.72	1.71
Double expenditure	1.68	1.68	1.64
Laspeyres	1.75	1.75	1.75
Paasche	1.70	1.61	1.52
Fisher's Ideal	1.72	1.67	1.63
Edgeworth	1.72	1.68	1.63

0-PATH AND 2-PATH

	$I=-27.3$	$I=-10.5$	$I=-5.7$
Exact indifference-defined index	5.68	5.43	5.15
Double expenditure	5.43	5.13	Not computed
Laspeyres	6.26	6.26	6.26
Paasche	4.86	3.87	3.29
Fisher's Ideal	5.52	4.92	4.54
Edgeworth	5.49	4.86	4.44

1-PATH AND 2-PATH

	$I=-27.3$	$I=-10.5$	$I=-5.7$
Exact indifference-defined index	3.38	3.05	3.03
Double expenditure	3.35	3.02	Not computed
Laspeyres	3.88	3.41	3.37
Paasche	3.13	2.75	2.48
Fisher's Ideal	3.30	3.06	2.89
Edgeworth	3.30	3.06	2.88

It seems that whenever Engel curves are given—or can be roughly interpolated—the double-expenditure method must be considered superior to the other known methods of computing price indices.

University of Norway

PROGRAM OF THE ANNECY MEETING, SEPTEMBER 11–15, 1937

The Seventh European Meeting of the Econometric Society was held at Annecy, Haute Savoie, France (not far from Geneva) from September 11 to 15, 1937. The following papers were given:

T. KOOPMANS, Linear Regression Analysis of Time Series.

P. DE WOLFF, The Movements of an Economic System under the Influence of Periodic External Disturbances.

R. W. JAMES, The Significance of the Characteristic Solutions of Mixed Difference and Differential Equations.

A. L. BOWLEY, Numerical Measurement of the Elasticity of Substitution.

A. L. BOWLEY, The Condition that Laspeyres' Index is Greater than Paasche's.

J. WISNIEWSKY, Consumption Units.

A. WALD, The Theory of Index Numbers.

J. M. FLEMING, Factors Determining the Amount of Money in Circulation.

F. SAMSON, Harvest Forecasts, Methods and Results.

J. TINBERGEN, The Influence of Labour Productivity Changes on the Trend of Employment.

T. KOOPMANS, On Trends Determined by Moving Averages.

The following persons attended: O. Anderson, Sofia; H. Bolza, Würzburg; A. L. Bowley, London; P. de Wolff, The Hague; P. Doyen, Le Mans; H. Duetschler, Zurich; L. Dumay, Paris; V. Edelberg, London; J. M. Fleming, Geneva; R. Gibrat, Paris; R. James, London; M. F. W. Joseph, Geneva; B. N. Kaul, Geneva; J. G. Koopmans, The Hague; T. Koopmans, Rotterdam; G. Lutfalla, Paris; H. Mendershausen, Geneva; J. W. Nixon, Geneva; J. M. I. Reitsma, Hengelo; E. Rothbarth, London; F. Samson, Geneva; K. Schlesinger, Vienna; H. Staehle, Geneva; J. Tinbergen, Geneva; F. J. C. van der Schalk, Bussum; E. C. van Dorp, Bloemendaal; A. Wald, Geneva; J. Wisniewski, Warsaw.

OFFICERS AND COUNCIL FOR 1938

The Council of the Econometric Society has elected the following officers for 1938: Arthur L. Bowley, President; Joseph A. Schumpeter, Vice-President; and Alfred Cowles 3rd, Secretary and Treasurer.

The Council members elected by the Fellows for terms to expire in December, 1940, are Ragnar Frisch, John Maynard Keynes, and F. Zeuthen. Council members whose terms expire in December, 1938, are Albert Aupetit, Arthur L. Bowley, and Wladyslaw Zawadzki. Those

whose terms expire in December, 1939, are Costantino Bresciani-Turroni, Irving Fisher, Charles F. Roos, and Joseph A. Schumpeter. Alfred Cowles 3rd, because of his position as Secretary and Treasurer, is an ex officio member of the Council.

ALFRED COWLES 3RD
Secretary

Colorado Springs

ELECTION OF FELLOWS, 1937

In accordance with the Constitution, the Fellows of the Society have elected the five new Fellows, whose names and partial bibliographies follow:

ALFRED COWLES 3RD, Cowles Commission for Research in Economics, Colorado Springs, Colorado, U.S.A.

"Can Stock Market Forecasters Forecast?" ECONOMETRICA, Vol. 1, July, 1933, pp. 309–324.

(With Edward N. Chapman), "A Statistical Study of Climate in Relation to Pulmonary Tuberculosis," *Journal of the American Statistical Association*, Vol. 30, Sept., 1935, pp. 517–536.

(With Herbert E. Jones), "Some A Posteriori Probabilities in Stock Market Action," ECONOMETRICA, Vol. 5, July, 1937, pp. 280–294.

Common Stock Indexes, 1871–1937, to be published February, 1938, about 475 pages.

J. R. HICKS, Gonville and Caius College, Cambridge, England.

"The Early History of Industrial Conciliation in England," *Economica*, Vol. 10, Mar., 1930, pp. 25–39.

"Edgeworth, Marshall, and the Indeterminateness of Wages," *Economic Journal*, Vol. 40, 1930, p. 215.

"The Theory of Uncertainty and Profit," *Economica*, Vol. 11, May, 1931, pp. 170–189.

"Marginal Productivity and the Principle of Variation," *Economica*, Vol. 12, Feb., 1932, pp. 79–88.

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JANUARY, 1938

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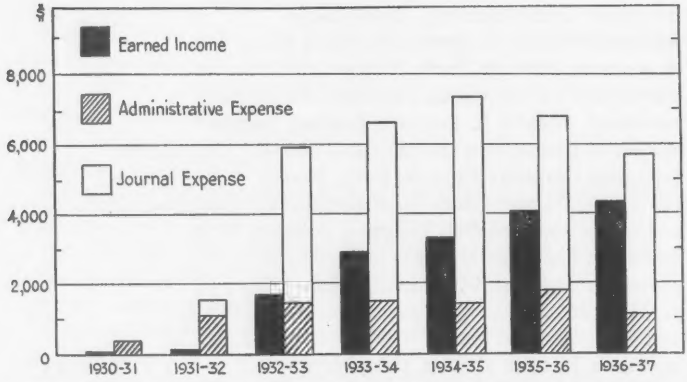
THE ECONOMETRIC SOCIETY

SUMMARY OF ACCOUNTS, 1930–1937



Years ending Sept. 30	EXPENSES			EARNED INCOME	DEFICIT
	ADMINISTRATION	JOURNAL	TOTAL		
1930–31	$ 405.44		$ 405.44	$ 27.00	$ 378.44
1931–22	1,069.86	$ 493.00	1,562.86	165.94	1,396.92
1933–33	1,486.50	4,456.26*	5,942.76*	1,774.19	4,168.57*
1933–34	1,508.82	5,054.98	6,563.80	2,866.68	3,697.12
1934–35	1,422.43	5,927.16	7,349.59	3,280.46	4,069.13
1935–36	1,809.32	4,972.27	6,781.59	4,083.73	2,697.86
1936–37	1,044.01	4,684.48	5,728.49	4,371.71	1,356.78

* Expenses for only the January, April and July numbers of ECONOMETRICA are included in 1932–33, due to the fact that the fiscal year ends September 30. If the October number had been added, the total expenses for the year would have been increased by approximately $700, to a figure about $100 in excess of the $6,563.80 reported for 1933–34.

The total deficit of $17,764.82 has been met in part by donations totalling $15,368.50. The balance of $2,396.32, together with a working balance of cash and other current net assets amounting to $603.68, is represented by notes of the Society outstanding to the amount of $3000. As security against this liability the Society, in addition to the cash balance, has a stock of approximately 440 complete files of ECONOMETRICA, Vols. 1–5, and about 2500 extra copies of various issues.

The deficit for 1936–37 was $1350 and new members and subscribers recently added will reduce this for the current year to approximately $1150. A 25 per cent increase in revenue, through the addition of 230 new members and subscribers, would balance the budget.

ALFRED COWLES 3RD
Treasurer

Colorado Springs

THE ECONOMETRIC SOCIETY

Alfred Cowles 3rd, Secretary
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I approve the election of all candidates for membership listed herewith. ☐

I approve the election of all candidates for membership listed herewith except those whose names I have indicated. ☐

Date .....Signature .....

To register your vote this ballot should be mailed, with the list of candidates attached, to the above address

THE ECONOMETRIC SOCIETY

LIST OF NOMINEES FOR ELECTION TO MEMBERSHIP

Compiled December, 1937

Each candidate listed below has been proposed by two members of the Society and nominated by a unanimous vote of the Nominating Committee of the Council. As an aid to members in determining the eligibility of candidates attention is called to the fact that the Constitution of the Econometric Society contains the provision that it shall operate as a completely disinterested, scientific organization without political, social, financial, or nationalistic bias. In order further to clarify the qualifications for membership, the officers and Council of the Society have recorded themselves as favoring the policy of considering as eligible for membership all serious students interested in the objectives of the Society, regardless of whether they have as yet achieved established reputations based on published works. Membership should be considered an honor, especially to young men, but it means merely that they have shown an understanding of the problems with which our Society deals.

Will members please suggest any corrections which should be made in the following names, titles, and addresses.

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